Lagrangian mean curvature flow in the complex projective plane

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A dissertation submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

of

University College London.

Department of Mathematics

University College London

January 17, 2022

I, Christopher G. Evans, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

In this thesis, we study Lagrangian mean curvature flow of monotone Lagrangians in two different settings, finding interesting and contrasting behaviour in each case.

First, we study the self-shrinking Clifford torus in $\mathbb{C}^2$. On the one hand, we find a family of C^k-small Hamiltonian deformations that force type II singularities to form. On the other hand, we find that any Hamiltonian deformation restricted to the unit sphere flows back to the self-shrinking Clifford torus after rescaling.

Second, we study Lagrangian mean curvature flow in Kähler–Einstein manifolds with positive Einstein constant. We show that monotone Lagrangians do not attain type I singularities under mean curvature flow, an analogue of a result of Wang [49].

Next, we investigate Lagrangian mean curvature flow of Vianna's exotic monotone tori ([47], [48]) in $\mathbb{CP}^2$. We define an $(S^1 \times \mathbb{Z}_2)$-equivariance, and we prove a Thomas–Yau-type result in this setting. We define a surgery procedure and show that any equivariant monotone Lagrangian torus exists for all time under mean curvature flow with surgery, undergoing at most a finite number of surgeries before converging to a minimal Clifford torus.

In particular, our result show that there does not exists a minimal equivariant Chekanov torus. Furthermore, we explicitly construct a monotone Clifford torus which has two finite-time singularities under mean curvature flow with surgery, becoming a Chekanov torus before eventually returning to become a Clifford torus again.

Impact Statement

Lagrangian mean curvature flow seeks to answer fundamental questions arising from mirror symmetry by means of a geometric flow. The process of finding and understanding special Lagrangian submanifolds and their deformations lies at the heart of the homological mirror symmetry and SYZ conjectures. These conjectures have ramifications for physics and string theory since Lagrangian submanifolds (or A-branes) are a proposed boundary condition for open strings.

Geometric flows have found many uses over the last 40 years, in mathematics and in other fields. In mathematics, Donaldson used Yang–Mills flow to find Hermitian Yang–Mills connections on stable bundles. Perhaps most notably, Perelman proved the Poincaré conjecture by use of Hamliton's Ricci flow. In physics, Bray solved the Riemannian Penrose conjecture by use of a geometric flow. In computer science, Geometric flows have found application in image processing.

In this thesis, we study Lagrangian mean curvature flow and its singular behaviour, particularly in the complex projective plane. Although the focus is on specific examples, the methods used are deliberately chosen to have a wider application. The ideas and tools developed are arguably just as important as the results, which show interesting, new and at times unexpected behaviour. We expect that the themes of the thesis and the methods used will be of use to anyone wishing to study Lagrangian mean curvature flow in Fano manifolds in the future.

Acknowledgements

I would like to thank my supervisors Jason Lotay and Felix Schulze for their constant support and encouragement. I count myself as very lucky to have had not only one excellent supervisor, but two.

Many thanks are owed to Jonny Evans for introducing me to the work of Vianna and encouraging me to study it in the context of mean curvature flow.

Thank you to Emily Maw for patiently explaining almost-toric fibrations to me. Thank you to Ben Lambert for many helpful talks over the years, and for struggling through some particularly heinous calculations to verify one of my results. Finally, thank you to Albert Wood, who not only explained in careful detail a great number of his results to me, but also read preliminary versions of this thesis and provided important feedback.

Thank you to my friends and partners-in-climb: Linda Bennett, Kwun Pan Chan, Thomas Körber. You made the last 4 years immensely more enjoyable.

This thesis couldn't have been completed without the constant support, encouragement and love of my partner, Kimberley. Thank you for everything.

This work was supported by the Leverhulme Trust Research Project Grant RPG-2016-174. The work of Jason Lotay and Felix Schulze, which contributed to Chapter 3, was also supported by the Leverhulme Trust Research Project Grant RPG-2016-174.

"What we know is a drop, what we don't know is an ocean."

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Chapter 1

Introduction

1.1 Historical remarks

1.1.1 Mirror symmetry and Lagrangian mean curvature flow

At the turn of the millennium, two revolutions were occurring in different halves of geometry. The first came in algebraic and symplectic geometry from a somewhat surprising source. In physics, attempts to quantize gravity had led to string theory. In this theory, particles gain a dimension to become 1-dimensional strings and this necessitates the enlargement of the background spacetime from 4 dimensions to 10. This is achieved by augmenting the usual 4-dimensional Minkowski space with tiny 6-dimensional manifolds, which for physical reasons known as supersymmetry must have the structure of Calabi–Yau 3-folds.

A pattern first observed by physicists while performing this quantization led to *mirror symmetry*, a conjectural identification of Calabi–Yau manifolds M with a mirror Calabi–Yau $\hat{M}$. Despite the lack of physical evidence in favour of string theory, or mathematical rigour in the physical derivation of the mirror symmetry, the series of conjectures spawned from this observation turned out to be remarkably prescient. They found initial success in enumerating various mirror pairs of Calabi–Yau manifolds with inverted Hodge numbers ($h^{1,1}(M) = h^{2,1}(\hat{M})$ and vice versa) [8] by a theory known as closed string mirror symmetry. A few years later, consideration of open string mirror symmetry led to two wide ranging and influential conjectures, known as Kontsevich's homological mirror symmetry or HMS [28] and the Strominger–Yau–Zaslow or SYZ conjecture [44]. In addition, evidence begun to mount that mirror symmetry existed in some form for a wider class of Kähler–Einstein manifolds in some form [23].

On the one hand, the SYZ conjecture posits a mirror relationship between special Lagrangian fibrations of M and $\hat{M}$. These special Lagrangian fibrations (i.e. fibrations of M by half-dimensional volume-minimising Lagrangian submanifolds) are largely conjectural and constructing them has been a difficult process. On the other hand, HMS relates the complex geometry of M to the symplectic geometry of $\hat{M}$ (and vice versa) by comparing the derived category of coherent sheaves $D^b\,\mathrm{Coh}(M)$ to the derived Fukaya category $D^b\mathscr{F}(\hat{M})$ of the mirror. Though coherent sheaves are well understood objects, the Fukaya category was not, and many parts of it are still poorly understood to this day. The Fukaya category $\mathscr{F}(M)$ has objects Lagrangian submanifolds of M and morphisms between them given by their intersections. Either way, it was fast becoming necessary to develop a deeper understanding of the Lagrangian structure of M, whether it was to find special Lagrangian fibrations or to better understand the Fukaya category.

The other revolution was occurring in Riemannian geometry and geometric analysis. The 1980s saw the introduction of two *geometric flows*, that is parabolic PDEs governing the evolution of some geometric structure. Hamilton introduced Ricci flow [20], an intrinsic flow given by evolving the metric g of a manifold according to its Ricci curvature Ric_g. Huisken introduced mean curvature flow [24], an extrinsic flow given by the gradient descent flow of the area functional; equivalently, submanifolds $F : M \to N$ allowed to evolve according to their mean curvature $\vec{H}$.

It had long been clear that analytic methods could prove to be tractable approaches to some of the most difficult problems in modern geometry. For instance, Yau's proof [53] [54] of the Calabi conjecture reduces the problem of finding Calabi–Yau metrics on a manifold M with $c_1(M) = 0$ to solving a fully non-linear, elliptic PDE known as a Monge–Ampère equation. Geometric flows first came to wider prominence with Perelman's proof [39] in 2003 of the Poincaré conjecture by Ricci flow, using a method identified by Hamilton [21] a few years earlier. The idea was to decompose 3-manifolds into canonical pieces by a process called Ricci flow with surgery. This presented great technical difficulties in understanding the nature of singularities of the flow, and the paper was in many ways well ahead of its time. The subsequent decade saw many attempts to further this work, both in the extension of Perelman's ideas to 4-manifolds and in the use of surgery in other geometric flows. Notably, in mean curvature flow, Huisken and Sinistrari [26] were able to decompose closed 2-convex hypersurfaces into canonical pieces using techniques similar to those used in Ricci flow.

Although seemingly quite disparate, these two revolutions are linked by the work of Smoczyk [41] and a conjecture of Thomas–Yau [46]. Smoczyk showed that the Lagrangian condition is preserved under mean curvature flow in Kähler–Einstein manifolds. Building on this, Thomas–Yau conjecture that under mean curvature flow, an almost-calibrated Lagrangian in a Calabi–Yau converges to the unique special Lagrangian in its Hamiltonian isotopy class. It has since become clear that singularities are in many cases inevitable in Lagrangian mean curvature flow, but adaptations of the Thomas–Yau conjecture were presented by Joyce [27]. Following the premise of the ideas of Thomas–Yau, Joyce focuses on the relationship between mirror symmetry, the Fukaya category and Lagrangian mean curvature flow. Put simply, his main premise is that by mirror symmetry, Lagrangian mean curvature flow should replicate on the symplectic side the features on the complex side.

1.1.2 Lagrangian mean curvature flow in Fano manifolds

Though the original Thomas–Yau conjecture and most of Joyce's paper focus on the Calabi–Yau case, mirror symmetry has been shown to work for Fano manifolds in a modified form and Joyce conjectures results in this setting in the final section of [27]. In this case, Thomas–Yau type-results require restriction to the class of monotone Lagrangians. From a symplectic point of view, monotone Lagrangians are natural to study since they capture aspects of the global topology of symplectic manifolds. From the point of view of Lagrangian mean curvature flow, monotone Lagrangians are arguably even more interesting. They are defined by the property that the mean curvature is exact, and hence they form a preserved class. Moreover, as we will show, they cannot attain type I singularities under mean curvature flow, thereby ruling out a large class of potential singularities.

The first Fano manifold of interest is the complex projective line or the 2-sphere

$\mathbb{CP}^1 = S^2$. Here all curves are Lagrangian and monotone Lagrangians divide the sphere in to equal area pieces. We understand Lagrangian mean curvature flow well in this setting: monotone Lagrangians exist for all time under the flow and converge to geodesic great circles in infinite time. This result mirrors that of the complex geometry of the mirror manifold to $\mathbb{CP}^1$: The mirror is a so-called Landau–Ginzburg model $(\mathbb{C}^*, z+1/z)$, and a technical but easily calculable result shows that there are only two interesting objects in the relevant category of coherent sheaves, each corresponding to the two spin structures on the equator.

A natural question is then to ask whether we can find similar results for higher dimension Fano manifolds. For instance, the mirror of $\mathbb{CP}^2$ is essentially no more complicated in its complex geometry than $\mathbb{CP}^1$. However, the symplectic geometry of $\mathbb{CP}^2$ is vastly more complicated.

Vianna ([47], [48]) found an infinite family of exotic monotone tori in the complex projective plane $\mathbb{CP}^2$. These tori are built inductively from a minimal monotone Lagrangian known as the Clifford torus by a series of Lagrangian surgeries called mutations. From a mean curvature flow point of view, these mutations resemble closely so-called Lawlor neck singularities, such as those observed by Neves [33] and constructed explicitly by Wood ([51], [52]).

The natural question then to ask is whether Lagrangian mean curvature flow collapses this family of exotic tori to the Clifford torus by a sequence of mean curvature flow surgeries. It is this idea that is the main premise of this paper, in particular the results of Chapter 4.

1.2 Summary of Results

We study mean curvature flow of Lagrangian tori in two different ambient manifolds: In Chapter 3, we study Clifford tori in $\mathbb{C}^2$, and in Chapter 4 we study Clifford and Chekanov tori in $\mathbb{CP}^2$. We prove contrasting results in each case, finding in each situations in which type II singularities are inevitable and also situations where Clifford tori are stable.

Section 3 is joint work with Lotay and Schulze, from the paper [16]. For Lagrangian mean curvature flow, type I singularities are not common. The work of Wang [49] shows that type I singularities do not occur for almost-calibrated Lagrangian mean curvature flows. The main goal of the paper [16] is to show that even for monotone Lagrangians, where type I singularities do exist, they are not the norm. We prove two results on the Lagrangian mean curvature flow of the self-shrinking Clifford torus $L_{\mathrm{Cl}} \in \mathbb{C}^2$:

Theorem 1.2.1. *Let $L_{Cl} \subset S^3(2) \subset \mathbb{C}^2$ be a self-shrinking Clifford torus. Then the following hold:*

1. ***Instability**: There exists an arbitrarily C^k-small Hamiltonian perturbation of L_{Cl} such that the flow does not flow back to the self-shrinking Clifford torus.*
2. ***Stability**: Any Hamiltonian deformation of L_{Cl} restricted to the 3-sphere $S^3(2)$ forms a type I singularity at the origin with type I blow-up given by L_{Cl}.*

We provide full definitions in the Chapters 2 and 3.

The first result extends earlier results of Groh–Schwarz–Smoczyk–Zehmisch [18] and Neves [34], where it was shown that the Clifford torus is unstable under sufficiently large Hamiltonian perturbations. Taken together, the two results answer a question of Neves [35, Question 7.4] asking for conditions on Lagrangian tori in $\mathbb{C}^2$ that guarantee convergence to a self-shrinking Clifford torus after rescaling at the singularity.

We also prove that L_{Cl} is locally unique as a self-shrinker for mean curvature flow (not just as a Lagrangian). We omit this discussion of this result from this thesis as it does not pertain to Lagrangian mean curvature flow.

In section 4, we study mean curvature flow in $\mathbb{CP}^2$. First of all, we show that type I singularities do not occur for monotone Lagrangian in Kähler–Einstein manifolds with positive Einstein constant:

Theorem 1.2.2. *Let $F_t : L^n \to M^{2n}$ be a monotone Lagrangian mean curvature flow in a Kähler–Einstein manifold M with Einstein constant $\kappa \neq 0$. Then F_t does not attain any type I singularities.*

This is the positive curvature equivalent of Wang's result [49] on type I singularities for almost-calibrated Lagrangian in Calabi–Yau manifolds.

For the remaining results, we need two mild extensions of existing well-known theorems. Since these results are independently useful, we state them here. First, we extend a result which dates back to Harvey–Lawson in the Calabi–Yau case:

Theorem 1.2.3. *Let Ω be a holomorphic volume form on an open subset $U \subset M$ of a Kähler–Einstein manifold. If L is a Lagrangian in U, then for any $X \in T_pL$ we have*

$$H(X) \cdot \Omega = d\theta(X) \cdot \Omega - i\nabla_X \Omega,$$

where H is the mean curvature 1-form and θ is the Lagrangian angle of L with respect to Ω.

Suppose now that M is an isometric toric manifold, that is to say there is an isometric Hamiltonian action of T^n on M^{2n}. Away from the singular points of the action, the level sets $\{L_\alpha\}$ are a Lagrangian fibration, and we can define Ω such that $\theta(L_\alpha) = 0$ for all α. Then for any $X = Y + JZ$ with $Y, Z \in T_pL$ we have

$$H(X) = d\theta(X) + H_{L_{\alpha(p)}}(Y),$$

where $H_{L_{\alpha(p)}}$ is the mean curvature 1-form of the unique Lagrangian L_α passing through p.

Second, we extend a result of Cieliebak–Goldstein [11] to include discs with corners, in the same way that the Gauss–Bonnet thoerem can be extended to include discs with corners. We use this result extensively in Chapter 4 to prove the main theorems. Furthermore, since it relies only on topological properties, it seems likely that it will be useful in many scenarios when considering Lagrangian mean curvature flow in Fano manifolds.

Extending the Maslov class μ of a disc D to a number $\tilde{\mu}$ accounting for turning angles at corners, we prove that

Theorem 1.2.4. *Let $L_1, \ldots, L_m$ be Lagrangian in a Kähler–Einstein manifold M and let*

$$u : (D, (\partial D_1, \ldots, \partial D_m)) \to (M, (L_1, \ldots, L_m)).$$

Then

$$\kappa \int_D \omega - \tilde{\mu}(D) = -\sum_i \int_{\partial D_i} H_{L_i}.$$

See Chapter 2 for definitions of terms used and proofs of these two theorems.

The main results of Chapter 4 concern monotone Lagrangian tori in $\mathbb{CP}^2$. We restrict to a subclass of Lagrangian tori L_γ which are generated from closed curves $\gamma \in \mathbb{C}$ by a particular $(\mathbb{Z}_2 \times S^1)$ rotation; we call these tori equivariant. This is a slightly stronger equivariance than has been considered previously in the literature, but since we already observe interesting and new behaviour in this class, it seems like a reasonable restriction. The main theorem of the chapter is the following:

Theorem 1.2.5. *Let L_γ be a monotone equivariant Lagrangian torus in $\mathbb{CP}^2$. Then under Lagrangian mean curvature flow with surgery, L_γ exists for all time with a finite number of surgeries and converges to a minimal Clifford torus as $t \to \infty$.*

We provide full definitions, including a precise definition of the equivariance and a definition of Lagrangian mean curvature flow with surgery, in Chapter 4.

There are two Hamiltonian isotopy classes of tori that can be realised as equivariant curves, namely the Clifford torus and the Chekanov torus, given respectively by curves γ containing and not containing the origin. The above result therefore implies that Chekanov tori undergo at least one surgery to become Clifford tori. We note that this implies that there is no equivariant minimal Chekanov torus, and we expect this to apply more generally.

The above result might seem to imply that a Clifford torus does not have singularities under Lagrangian mean curvature flow. However, this is not the case:

Theorem 1.2.6. *There exists a Clifford torus L_γ in $\mathbb{CP}^2$ such that under mean curvature flow L_γ has a finite-time singularity and surgery at the singularity makes L_γ a Chekanov torus.*

Thus the behaviour we observe is of a cyclical nature: a Clifford torus can collapse to become Chekanov torus, which then exists for some time after surgery before collapsing to a Clifford torus again. This process can repeat any finite number of times before eventually becoming a stable Clifford torus. We categorise this behaviour in the proof of

Theorem 1.2.5 by observing that a certain intersection number decreases under the flow with surgery.

1.3 Notations and conventions

Except where explicitly notated otherwise, we use the following conventions throughout this thesis.

We consider mean curvature flow of Lagrangians $F_t : L \to M$ in $(2n)$-dimensional Kähler–Einstein manifolds M, with Einstein constant κ. We will drop the subscript t when the meaning is clear. We will frequently abuse notation by conflating a manifold with its embedded image (writing for instance $L_t = L = F(L) = F_t(L)$), and conflate vectors with pushforwards and forms with pullbacks, as is standard in the literature. We extend these conventions to curves $\gamma(s) \in \mathbb{C}$.

By default, Lagrangians are assumed to be embedded, orientable, and all curves $\gamma(s) \in \mathbb{C}$ are assumed to be closed and simple. The main exceptions to the above are cones/real projective planes/lines through the origin considered in Chapter 4, and the minimal immersed Lagrangians considered in Section 4.6.

Chapter 2

Preliminaries

2.1 Lagrangians in Kähler–Einstein manifolds

Let M^{2n} be a manifold of even dimension. We recall the following definitions:

Definition 2.1.1.

1. A Riemannian metric g is a positive-definite symmetric 2-tensor on M. (M,g) is called a Riemannian manifold.

2. A symplectic form ω is a closed, non-degenerate 2-form on M. (M,ω) is called a symplectic manifold.

3. An almost-complex structure J is a (1,1)-tensor on M satisfying $J^2 = -1$. (M,J) is called an almost-complex manifold.

4. A triple of structures (g,ω,J) is called compatible if $g(X,Y) = \omega(X,JY)$. (M,g,ω,J) is called almost-Kähler.

5. In addition, if J is integrable, or equivalently $\nabla J = 0$ where ∇ is the Levi–Civita connection of g, (M,g,ω,J) is called Kähler.

6. In addition, if $\mathrm{Ric}_g = \kappa g$ for some constant κ, (M,g,ω,J) is called Kähler–Einstein with Einstein constant κ.

7. In addition, if $\kappa = 0$, we will call M a Calabi–Yau manifold, and if $\kappa > 0$ we will call M a Fano manifold.[1]

In the entirety of this thesis, M is assumed to be Kähler–Einstein with constant κ. The following two examples constitute the main focus of Chapters 3 and 4 respectively.

Example 2.1.2. Let

$$\mathbb{C}^n = \{(z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n) : x_j, y_j \in \mathbb{R}\}$$

with

$$\begin{aligned} g &= dx_1^2 + dy_1^2 + \cdots + dx_n^2 + dy_n^2 \\ \omega &= dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n \\ J(\partial_{x_j}) &= \partial_{y_j}, \quad J(\partial_{y_j}) = -\partial_{x_j} \end{aligned}$$

Then $\mathbb{C}^n$ satisfies all the above definitions with $\kappa = 0$.

Example 2.1.3. Let $\mathbb{CP}^n$ be complex projective space, i.e. the quotient of complex n-space $\mathbb{C}^n$ by the standard action of $\mathbb{C}^*$:

$$(z_0, \dots, z_n) \sim (wz_0 : \cdots : wz_n), \quad w \in \mathbb{C}^*.$$

Alternately, $\mathbb{CP}^n$ is the quotient of the unit sphere $S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ under the identification

$$(z_0, \dots, z_n) \sim (e^{i\alpha} z_0 : \cdots : e^{i\alpha} z_n), \quad e^{i\alpha} \in S^1,$$

known as the Hopf fibration. The quotient induces a Riemannian metric on $\mathbb{CP}^n$ from the round metric on the sphere as a Riemannian submersion called the Fubini–Study metric. Similarly, the quotient from $\mathbb{C}^n$ induces a symplectic form (the Fubini–Study

[1] Neither of these definitions is standard. For instance, in the Calabi–Yau case, various authors require some combination of additional conditions such as compact, simply connected or with holonomy exactly equal to SU(2). For Fano manifolds, a metric is typically not specified, instead requiring that the anti-canonical bundle K_M^* is ample. For the purposes of Lagrangian mean curvature flow, the definitions we give make the most sense: A Riemannian metric is always specified by default and the most important factor is the Ricci curvature of the ambient space.

form) and an almost complex structure. Compatibility follows from compatibility of the forms in Example 2.1.2. Indeed, $\mathbb{CP}^n$ is Kähler–Einstein with constant $2(n+1)$.

Definition 2.1.4. Let M^{2n} be a symplectic manifold. Then a n-dimensional submanifold $F : L^n \to M^{2n}$ is called Lagrangian if $F^*\omega = 0$.

As is standard in the literature, we will often abuse notation by conflating a manifold L with its embedding/immersion $F(L)$. We extend this abuse of notation in the natural way to vectors/tensors and their pushforwards/pullbacks.

2.1.1 The Lagrangian angle and mean curvature

In this section, we describe how one can characterise the mean curvature of Lagrangian submanifolds using aspects of their symplectic geometry. In the Calabi–Yau sense, ideas in this vein date back to Harvey–Lawson [22] and [36]. The material is all standard, but we emphasise the perspective that the usual calculations apply locally and can be modified to account for non-Ricci-flat manifolds.

Consider the holomorphic volume form

$$\Omega = dz_1 \wedge \cdots \wedge dz_n$$

on $\mathbb{C}^n$. The space of Lagrangian subspaces $\mathscr{L}(n)$ is isomorphic to $\mathrm{U}(n)/\mathrm{O}(n)$, so any Lagrangian subspace Λ may be written as $A \cdot \mathrm{span}\{x_1, \ldots, x_n\}$ for some unitary matrix A, unique up to orthogonal transformations. Thus we have that

$$|\Omega_\Lambda| = |A| \cdot \Omega(\partial_{x_1}, \ldots, \partial_{x_n}) = 1$$

and hence

$$\Omega_\Lambda = e^{i\theta}\,\mathrm{vol}_\Lambda$$

for some θ. Now let Ω be a non-vanishing holomorphic volume form on some open subset $U \subset M$, i.e. a section of the canonical bundle K_U. Then there exist complex coordinates $z_1, \ldots, z_n$ on some subset $V \subset U$ and a holomorphic function f on V such that

$$\Omega = f\,dz_1 \wedge \cdots \wedge dz_n.$$

Furthermore, by Darboux's theorem, we can choose V small enough that V is symplectomorphic to $\mathbb{C}^n$ with the standard structure, and hence we can conclude that the above extends to any Lagrangian L in U, i.e. we have

$$\Omega_L = e^{i\theta}\,\mathrm{vol}_L$$

for some function $\theta : L \to \mathbb{R}/2\pi$.

In the case that M is Calabi–Yau, one can find a global non-vanishing holomorphic volume form Ω, so the above applies to any Lagrangian L in M. In this case, we call θ the Lagrangian angle of L. For more general Kähler manifolds with local holomorphic volume forms, we call θ the Lagrangian angle of L relative to Ω. Furthermore, we call Lagrangians L with $\theta = 0$ special Lagrangian (relative to Ω).

We now investigate the mean curvature of Lagrangian submanifolds. Recall that a Riemannian submanifold $F : L \to (M,g)$ of a Riemannian manifold has second fundamental form

$$A(X,Y) = \overline{\nabla}_X Y - \nabla_X Y,$$

a symmetric 2-form with values in the normal bundle, where $\overline{\nabla}$ is the Levi–Civita connection of the ambient metric g, and ∇ is Levi–Civita connection of the induced metric F^*g. Since

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = \nabla_X Y - \nabla_Y X = [X,Y],$$

we immediately see that $A(X,Y) = A(Y,X)$, i.e. A is symmetric and hence is tensorial. The mean curvature is then defined to be the trace of the second fundamental form

$$\vec{H} = \mathrm{trace} A.$$

Note that both A and $\vec{H}$ lie in the normal bundle to L. Lagrangian submanifolds in Kähler manifolds have the property that J is an isometry between the tangent and normal bundles of L. Thus we have isometries

$$NL \cong TL \cong T^*L.$$

Furthermore, the form

$$H(X) = \omega(X, \vec{H})$$

is zero restricted to NL since L is Lagrangian and $\vec{H}$ is normal, hence it is appropriate to consider H a 1-form on L, called the mean curvature 1-form. Furthermore, in the case that M is Kähler–Einstein, H is closed (see for instance [43], though we prove stronger results in various settings later in this thesis.) This raises the possibility that H could be exact, and as the following proposition shows, this holds in a Calabi–Yau whenever the Lagrangian angle is a function on L.

Since the details of the proof are important later, we present a proof here, following the method found in Oh [38] or Thomas–Yau [46]. We stress that the proof relies on the Calabi–Yau specific fact that Ω is parallel.

Proposition 2.1.5. *Let L be an oriented Lagrangian in a Calabi–Yau manifold M with mean curvature 1-form H. Then $H = d\theta$, where θ is the Lagrangian angle.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for T_pL and extend to a local frame such that $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ is a basis for TM near p. We may choose this basis such that $\nabla_{e_i} e_j$ vanishes for all i, j at p. In this basis, we have that

$$\Omega_L = \bigwedge_i (e_i^* + i(Je_i)^*)\,,$$

where e_i^* is the dual of e_i, and hence

$$\Omega = e^{-i\theta} \bigwedge_i (e_i^* + i(Je_i)^*)$$

Since $\overline{\nabla}\Omega = 0$ in Calabi–Yau manifolds, it follows that

$$\begin{aligned} 0 &= \overline{\nabla}_X \Omega \\ &= -id\theta(X) \cdot \Omega + e^{-i\theta} \sum_j (e_1^* + i(Je_1)^*) \wedge \cdots \wedge \overline{\nabla}_X (e_j^* + i(Je_j)^*) \wedge \cdots \wedge (e_n^* + i(Je_n)^*). \end{aligned}$$

Since any terms of $\overline{\nabla}_X (e_j^* + i(Je_j)^*)$ which are not a scalar multiple of $(e_j^* + i(Je_j)^*)$ vanish in the wedge product, we may rewrite the second term as

$$\sum_j \left(\overline{\nabla}_X (e_j^* + i(Je_j)^*) \left(\frac{1}{2}(e_j - iJe_j) \right) \right) \cdot \Omega$$

So it follows that

$$\begin{aligned}
id\theta(X) &= \frac{1}{2}\sum_j \left(\overline{\nabla}_X \left((e_j^* + i(Je_j)^*) \left(e_j - iJe_j\right)\right) - (e_j^* + i(Je_j)^*)(\overline{\nabla}_X (e_j + iJe_j))\right) \\
&= -\frac{1}{2}\sum_j (e_j^* + i(Je_j)^*)(\overline{\nabla}_X (e_j + iJe_j)) \\
&= -\frac{i}{2}\sum_j \left(e_j^*(J\overline{\nabla}_X e_j) + (Je_j)^*(\overline{\nabla}_X e_j)\right) \\
&= -i\sum_j g(e_j, J\overline{\nabla}_X e_j),
\end{aligned}$$

where we have used the fact that $(e_j^* + i(Je_j)^*)\left(\frac{1}{2}(e_j - iJe_j)\right) = 1$ in the first two lines, and have used the assumption that $\nabla_{e_i} e_j$ vanishes at p on the fourth line. Thus it suffices to show that

$$H(X) = \sum_j g(e_j, J\overline{\nabla}_X e_j) = \sum_j \omega(e_j, \overline{\nabla}_X e_j).$$

We calculate that

$$H(X) = \omega(X, \vec{H}) = \sum_j \omega\left(X, \overline{\nabla}_{e_j} e_j\right) = \sum_j \omega(e_j, \overline{\nabla}_{e_j} X) = \sum_j \omega(e_j, \overline{\nabla}_X e_j),$$

since $\omega(e_j, X) = 0$ and we may assume that $[X, e_j]$ vanishes. This completes the proof.

□

Note that the above proof implies that for any holomorphic volume form Ω defining a Lagrangian angle by $\Omega_L = e^{i\theta}\,\mathrm{vol}_L$, we have

$$H(X) \cdot \Omega = d\theta(X) \cdot \Omega - i\nabla_X \Omega$$

for any $X \in T_pL$. This demonstrates the first part of Theorem 1.2.3. For the second part, we first construct a holomorphic volume form Ω from a Lagrangian fibration.

Let $\{L_\alpha\}_{\alpha \in I}$ be a Lagrangian torus fibration of a subset $U \in M$. For each α, define a holomorphic volume form Ω_{L_α} along L_α, i.e. a unit section of the canonical bundle

$K_{M|_{L_\alpha}}$, by

$$\Omega_L(X_1,\ldots,X_n) = \mathrm{vol}_L(X_1,\ldots,X_n),$$
$$\Omega_L(JX_1,X_2,\ldots,X_n) = i\,\mathrm{vol}_L(X_1,\ldots,X_n), \quad \text{etc.}$$

for tangent vectors $X_i \in T_pL$. Now let $x \in U$. There is a unique $\alpha(x) \in I$ such that $x \in L_{\alpha(x)}$, so we define a section of $K_{M|_U}$ by

$$\Omega(x)(X_1,\ldots,X_n) = \Omega_{L_{\alpha(x)}}(X_1,\ldots,X_n),$$

for $X_i \in T_xM$. We call Ω a relative holomorphic volume form (to the fibration L_α).

In contrast to the Calabi–Yau case where Ω was always parallel, the volume form defined here is in general not parallel. In [30], the form Ω_L is differentiated in tangent and normal directions. For tangent vector fields $X \in \Gamma(TL)$ we have

$$\bar{\nabla}_X \Omega_L = iH_L(X)\,\Omega_L \tag{2.1.1}$$

where H_L is the mean curvature 1-form on L. On the other hand, if $JY = \frac{\partial A_s}{\partial s}|_{s=0}$ is the normal vector field corresponding to a 1-parameter family $A_s : L \to M$ of Lagrangians immersions then the normal derivative is

$$\bar{\nabla}_{JY} \Omega_L = -i\,\mathrm{div}_L(Y)\,\Omega_L. \tag{2.1.2}$$

Now suppose the fibration $\{L_\alpha\}$ are the level sets of a moment map for an isometric Hamiltonian T^n-action on U. Since the action is an isometry, any vector field $\tilde{X}$ generated by the subgroup $\exp(tX)$ of T^n has $\mathscr{L}_{\tilde{X}}\,\mathrm{vol}_{L_\alpha} = \mathrm{div}_{L_\alpha}(\tilde{X})\,\mathrm{vol}_{L_\alpha} = 0$. Furthermore, since the action is Hamiltonian, $J\tilde{X}$ is a normal vector field corresponding to a 1-parameter family of Lagrangian immersions. So we have shown the second part of Theorem 1.2.3.

2.2 Lagrangian mean curvature flow

The fact that H is closed seems to imply that the Lagrangian condition is preserved under mean curvature flow since

$$\frac{\partial}{\partial t}F = \vec{H}$$

implies

$$\frac{\partial}{\partial t}F^*\omega = d\left(\omega\left(\frac{\partial}{\partial t}F,\cdot\right)\right) = dH = 0.$$

This is clearly a necessary condition for the Lagrangian condition to be preserved, but is not a priori sufficient. However, this premise can be integrated up to show:

Theorem 2.2.1 (Smoczyk [41])**.** *In Kähler–Einstein manifolds, the Lagrangian condition is preserved for closed submanifolds under mean curvature flow.*

We now present some of the basic facts of Lagrangian mean curvature flow, beginning with evolution equations. The following calculation appears in Thomas–Yau [46], but the results were known by Oh [38] and Smoczyk [42]. For any holomorphic volume form Ω defining a Lagrangian angle by $\Omega_L = e^{i\theta}\,\mathrm{vol}_L$, we have

$$\begin{aligned} i\frac{\partial}{\partial t}\theta e^{i\theta}\,\mathrm{vol}_L + e^{i\theta}\frac{\partial}{\partial t}\mathrm{vol}_L =& \frac{\partial}{\partial t}\left(e^{i\theta}\,\mathrm{vol}_L\right) = \mathscr{L}_V\Omega_L = d\left(V \lrcorner \Omega_L\right) = -i\,d\left(e^{i\theta}(JV)\lrcorner\,\mathrm{vol}_L\right) \\ =& e^{i\theta}d\theta\wedge(JV\lrcorner\,\mathrm{vol}_L) - ie^{i\theta}d^\dagger(JV)\,\mathrm{vol}_L \end{aligned}$$

Under mean curvature flow, i.e. $V = \vec{H} = J\nabla\theta$ one recovers from this the evolution equations

$$\begin{aligned} \frac{\partial}{\partial t}\theta &= d^\dagger d\theta = \Delta\theta \\ \frac{\partial}{\partial t}\mathrm{vol}_L &= -|\vec{H}|^2\,\mathrm{vol}_L \end{aligned} \tag{2.2.1}$$

The mean curvature 1-form H satisfies the evolution equation

$$\frac{\partial}{\partial t}H = dd^\dagger H + \kappa H, \tag{2.2.2}$$

where κ is the Einstein constant, i.e. $\rho = \kappa\omega$. It is clear then that the cohomology class $[He^{-\kappa t}]$ is preserved under the flow. In particular, H exact is preserved.

We now highlight a few particular subclasses of Lagrangians in Calabi–Yau manifolds, each preserved under mean curvature flow. In order of inclusion:

1. **Special Lagrangian**s: Lagrangian submanifolds with $\theta = 0$.
2. **Minimal Lagrangians**: Lagrangian submanifolds with $\vec{H} = 0$. All special Lagrangians are minimal, and all minimal Lagrangians are special up to a choice of phase.
3. **Almost-calibrated Lagrangians**: Lagrangian submanifolds where $\cos\theta > \varepsilon > 0$ everywhere, up to a choice of phase.
4. **Zero-Maslov Lagrangians**: Lagrangian submanifolds with $\theta : L \to \mathbb{R}$ a real-valued function (i.e. not circle valued).

The final subclass of interest is that of monotone Lagrangians. Recall that the space of Lagrangian subspaces $\mathscr{L}(n)$ in $\mathbb{R}^{2n}$ is isomorphic to $U(n)/O(n)$, and hence $\det^2$ induces an isomorphism from $\mu : \pi_1(\mathscr{L}(n)) \to \mathbb{Z}$, called the Maslov index. The Maslov class of a disc is defined to be the Maslov index of the boundary under any local trivialisation. Then we have the following theorem of Cieliebak–Goldstein [11], which is fundamental to the rest of this thesis:

Theorem 2.2.2 (Cieliebak–Goldstein)**.** *In a Kähler–Einstein manifold M with Einstein constant κ, the mean curvature 1-form H of a Lagrangian $F : L \to M$ is related to the Maslov class μ of a disc $u : (D, \partial D) \to (M, L)$ by* [2]

$$\kappa \int_D \omega - \pi\mu(D) = -\int_{\partial D} H. \tag{2.2.3}$$

We call a Lagrangian submanifold monotone if for any disc $u : (D, \partial D) \to (M, L)$,

$$\int_D \omega = c\mu(D), \tag{2.2.4}$$

[2] Here and throughout the rest of this thesis, we abuse notation by conflating forms with their pullbacks and curves in the image of a Lagrangian with their pre-image. For instance, in (2.2.3),

$$\int_D \omega = \int_D u^* \omega, \quad \int_{\partial D} H = \int_{F^{-1}(u(\partial D))} H.$$

for a constant c dependent on M and L but not u. We call a disc u Maslov m if the $\mu(u) = m$. In the case of a monotone Lagrangian in an exact Calabi–Yau manifold (i.e. $\omega = d\lambda$), and in view of (2.2.3), we see that (2.2.4) is equivalent to

$$\int_{\partial D} \lambda = \int_D \omega = c\mu(D) = \frac{c}{\pi}\int_{\partial D} H = \frac{c}{\pi}\int_{\partial D} d\theta,$$

hence in the literature for Lagrangian mean curvature flow where the Calabi–Yau case (specifically $\mathbb{C}^n$) is frequently the primary focus, the definition of monotone is often taken as

$$[\lambda] = C[d\theta].$$

Remark 2.2.3. The Cieliebak–Goldstein formula is a generalisation of the Gauss-Bonnet formula. When M is a surface, ω is the Riemannian volume form on M, and so M Einstein implies that

$$\int_D K \operatorname{vol}(D) = \kappa \int_D \omega,$$

where K is the Gauss curvature. Moreover, all curves are Lagrangian so

$$\int_{\partial D} k_g \operatorname{vol}(\partial \mathrm{D}) = \int_{\partial D} H$$

where k is the geodesic curvature. The Euler characteristic of a disc is 1, and the Maslov class of a holomorphic disc in a symplectic surface is 1 by definition.

The above remark helps to motivate a mild generalisation of the Cieliebak–Goldstein formula to include J-holomorphic polygons with boundary on multiple intersecting Lagrangians, comparable to generalising Gauss–Bonnet with a smooth boundary to a piecewise-smooth boundary with corners and turning angles.

Theorem 2.2.4. *Let $L_1,\dots,L_m$ be Lagrangian in M and let*

$$u : (D, (\partial D_1, \dots, \partial D_m)) \to (M, (L_1, \dots, L_m))$$

denote a map from the unit disc with m marked points p_i on the boundary to M, mapping

p_i to $L_i \cap L_{i-1}$ and mapping the arc ∂D_i from p_i to p_{i+1} to L_i. Then

$$\kappa \int_D \omega - \pi \tilde{\mu}(D) = -\sum_i \int_{\partial D_i} H_{L_i}, \tag{2.2.5}$$

where $\tilde{\mu}$ is the Maslov class of $u : (D, (\partial D_1, \ldots, \partial D_m)) \to (M, (L_1, \ldots, L_m))$, defined in the proof below.

Proof. We first redefine the Maslov class in a way more suited to proving the theorem. Consider a Lagrangian subspace L of a complex vector space $\mathbb{C}^n$. Let $\Lambda^{(n,0)}\mathbb{C}^n$ be the space of $(n,0)$-forms on $\mathbb{C}^n$, i.e. the space of holomorphic volume forms. Since L is a Lagrangian subspace, there is an element $\tau(L)$ in $\Lambda^{(n,0)}\mathbb{C}^n$ of unit length which restricts to give a real volume form on L. One observes this immediately for the Lagrangian subspace $L_0 = \text{span}\{x_1, \ldots, x_n\}$ with $\tau(L_0) = dz_1 \wedge \cdots dz_n$, and since any Lagrangian subspace $L = A \cdot L_0$ for some $A \in \mathrm{U}(n)$, we have $\tau(L) = A \cdot \tau(L_0)$ where we let $\mathrm{U}(n)$ act on holomorphic $(n,0)$-forms as $\det(A)$. The map A is unique up to the action of $\mathrm{O}(n)$, so $\tau(L)$ is unique up to sign. Thus we obtain a unique element

$$\tau^2(L) \in K^2(\mathbb{C}^n) = \Lambda^{(n,0)}\mathbb{C}^n \otimes \Lambda^{(n,0)}\mathbb{C}^n.$$

Now let L be a Lagrangian in a Kähler manifold M^{2n}. Treating the tangent space of L as a Lagrangian subspace of $\mathbb{C}^n$, as above we have a unit-length section

$$\tau_L^2 : L \to K^2(M),$$

where $K^2(M)$ is the square of the canonical bundle $K(M) = \Lambda^{(n,0)}T^*M$ of M.

Let $u : (D, \partial D) \to (M, L)$ be a smooth map from the unit disc to M with boundary in L. Since $H^2(D) = 0$, $u^*K(M)$ is trivial so there exists a unit-length section τ_u of $K(M)$ over $u(D)$. On the boundary, there exists an S^1-valued function $e^{i\alpha} : \partial D \to S^1$ such that

$$\tau_L^2 = e^{i\alpha}\tau_u^2.$$

We define the Maslov class as minus the winding number of $e^{i\alpha}$, i.e.

$$\mu(D) = \frac{-1}{2\pi}\int_{\partial D} d\alpha.$$

This definition agrees with the usual definition of Maslov class, as can be verified by checking it satisfies the standard axiomatic description.

Now, suppose we have Lagrangians $L_1, \dots, L_m$ in M. We have, as above, sections $\tau^2_{L_i}$. For any $u : (D, (\partial D_1, \dots, \partial D_m)) \to (M, (L_1, \dots, L_m))$, we obtain as above S^1-valued functions $e^{i\alpha_i} : \partial D_i \to S^1$ with

$$\tau^2_{L_i} = e^{i\alpha_i}\tau^2_u.$$

We define the Maslov number $\tilde{\mu}(D)$ (here no longer integer-valued) by

$$\tilde{\mu}(D) = \frac{-1}{2\pi}\sum_i \int_{\partial D_i} d\alpha_i,$$

which we note is well-defined.

With the definitions out of the way, we can prove the theorem. For simplicity, we prove for $m = 1$, i.e. the case where we have a Lagrangian $L = L_1$ with a single self-intersection. This suffices for all the proofs later in the thesis, but the more general result stated above follows a similar argument. We sketch the proof, a more detailed discussion can be found in [11].

As in Cieliebak–Goldstein, we have that τ^2_L defines an imaginary-valued connection 1-form η_L by $\nabla\tau^2_L = \eta_L \otimes \tau^2_L$. It is a classical result due to Oh [37] that $\eta_L = 2iH$.

Similarly, for the section τ^2_u, we have an imaginary-valued connection 1-form η_u defined by $\nabla\tau^2_u = \eta_u \otimes \tau^2_u$. We have that $d\eta_u = -2R_K$, where R_K is the curvature of $K(M)$, which in the Kähler–Einstein case implies that $d\eta_u = 2i\rho = 2i\kappa\omega$.

The connection 1-forms η_u and η_L are related by

$$\eta_L = \eta_u + i\,d\alpha,$$

so

$$\int_{\partial D} H = \int_{\partial D} -\frac{i}{2}\eta_L = \int_{\partial D} -\frac{i}{2}\eta_u + \frac{1}{2}\int_{\partial D} d\alpha = \kappa\int_D \omega - \pi\tilde{\mu}(D).$$

□

Let us now consider the implications of the Cieliebak–Goldstein formula for Lagrangian mean curvature flow. From (2.2.3) and the evolution equation (2.2.2) we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\int_D \omega = -\frac{1}{\kappa}\int_{\partial D} dd^{\dagger}H + \kappa H &= -\int_{\partial D} H \\ &= \kappa \int_{\partial D} \omega - \pi\mu(D). \end{aligned} \tag{2.2.6}$$

L is monotone when $\mu(D)$ is proportional to $\int_D \omega$, so we note two immediate corollaries for $\kappa \neq 0$.

Corollary 2.2.5. *Let L be a Lagrangian in a Kähler–Einstein manifold with $\kappa \neq 0$. H is exact if and only if L is monotone with monotone constant π/κ.*

Corollary 2.2.6. *Monotone Lagrangians are preserved under mean curvature flow. When $\kappa \neq 0$, the monotone constant π/κ is invariant under the flow.*

Proof. The result has been shown already for $\kappa = 0$. For $\kappa \neq 0$, the result follows from Corollary 2.2.5 and the fact that exactness of H is preserved by equation (2.2.2). □

To illustrate the theory so far, we consider the best understood example of Lagrangian mean curvature flow in non-Ricci-flat manifolds.

Consider the two sphere $S^2 = \mathbb{CP}^1$ with the standard Kähler metric and let γ be an embedded closed curve in S^2. Then there are, up to reparametrisation, exactly two J-holomorphic discs $u_1, u_2 : D \to S^2$ with $u_i(\partial D) = \gamma$. We have that γ is monotone when

$$\int_D u_1^*\omega = \int_D u_2^*\omega,$$

where ω is the standard Fubini–Study form on $\mathbb{CP}^1 = S^2$, i.e. when γ divides S^2 into two pieces of equal area. Then we have two behaviours:

Proposition 2.2.7.

1. *If γ is not monotone, γ attains a type I singularity in finite time with blow-up a self-shrinking circle.*

2. *If γ is monotone, mean curvature flow exists for all time and converges in infinite time to a great circle.*

Proof. Recall Grayson's theorem [17]: curve-shortening flow in surfaces either attains finite-time singularities with type I blow-up a shrinking circle, or exists for all times and converges to a geodesic. This the result follows from (2.2.6) in both cases. □

2.3 Singularities in Lagrangian mean curvature flow

In mean curvature flow, singularities are studied and classified by analysing their behaviour at the singular time using a procedure called a blow-up. A solution $F_t : L \to M$ to the mean curvature flow equation

$$\frac{\partial}{\partial t} F_t = \vec{H},$$

always exists on some maximal time interval $[0, T)$, though to guarantee uniqueness one has to make modifications to account for reparametrisations. If $T < \infty$, then it can further be shown that

$$\limsup_{t \to T} \max_{L_t} |A|^2 = \infty,$$

which Smoczyk [43] comments is a folklore result for higher codimension mean curvature flow: the proof follows the same lines as the proof for codimension 1, where the result can be achieved relatively simply by considering the evolution equations for $|\nabla^m A|^2$, and showing that bounds on $|A|^2$ imply bounds on $|\nabla^m A|^2$ for all $m \geq 0$. Bounds on all derivative of $|A|^2$ imply smooth convergence to a limiting immersion $F_T : L \to M$ and hence short-time existence implies T is not the maximal time.

Singular behaviour is common in mean curvature flow. For instance, any compact initial condition $F_0 : L \to \mathbb{R}^n$ achieves a singularity at a finite time T. Singularities are classified into two types based on the rate of blow-up of $|A|^2$. If

$$\sup_{L_t} |A|^2 \leq C(T - t)^{-1},$$

for some constant C, we call the singularity type I, and if no such bound exists, we call it type II. The primary reason for this distinction is the following: for $\lambda > 0$, $\tilde{x} = \lambda(x - x_0)$, $\tilde{t} = \lambda^2(t - t_0)$,

$$\tilde{F}_{\tilde{t}}^{\lambda} := \lambda \left(F_{t_0 + \lambda^{-2}\tilde{t}} - x_0 \right)$$

is a mean curvature flow, called a parabolic rescaling. Using Huisken's monotonicity formula, one can show that any sequence of parabolic rescalings $F_{\tilde{t}}^{\lambda_i}$ with $\lambda_i \to \infty$ at a type I singularity $(x_0, t_0) = (x, T)$ of the flow converges subsequentially to a smooth limiting flow $F_{\tilde{t}}^{\infty}$, called a type I blow-up (possibly not unique), with the property that

$$\vec{H} = \frac{x^{\perp}}{2\tilde{t}}.$$

Solitons of mean curvature flow of this type are called *self-shrinkers* since they flow by homotheties. If the singularity is instead type II, one still can find a weak limit to parabolic rescalings, though now the limiting flow is a *Brakke flow* [6]: a flow of rectifiable varifolds rather than smooth manifolds. We also call this limit a type I blow-up, even though the singularity is type II.

Consider embedded hypersurface mean curvature flow. In this case, type I singularities are conjecturally generic. The simplest case is curve shortening flow in the plane (i.e. the case of curves $\gamma_t : S^1 \to \mathbb{R}^2$). Here, all curves attain type I singularities with type I blow-up a self-shrinking circle in finite time. This kind of regularity does not exist in higher dimensions, even though we still expect type I singularities to be the norm. Type II singularities do exist, but tend to occur as degenerate cases. The prototypical example of this behaviour is for a dumbbell in $\mathbb{R}^3$, i.e. a smoothing of a connect sum of two spheres S_1, S_2 with radii $r_1 \leq r_2$ by a cylindrical neck of radius s. Here, three behaviours may be observed. Firstly, if the neck is narrow enough, the neck shrinks to form a type I singularity with type I blow-up given by a self-shrinking cylinder. Secondly, if s is large enough compared to r_1, the sphere may shrink before the neck collapses and prevent a cylindrical singularity forming, before eventually collapsing to a self-shrinking sphere. The third option is the degenerate case when $r_1 < r_2$. If r_1 is chosen carefully with respect to s, one can obtain a behaviour where S_1 shrinks at precisely the same time that the neck shrinks, leading to a type II singularity.

This behaviour is to be expected according to the work of Huisken–Sinestrari [26]. They show that 2-convex hypersurfaces in $\mathbb{R}^n$, $n \geq 4$ attain singularities of the three types above, and go further in providing surgery procedures for each of the three cases. In this way, they can decompose 2-convex hypersurfaces into constituent parts by mean curvature flow with surgery.

In contrast to the hypersurface case, type I singularities are rare in Lagrangian mean curvature flow, and type II singularities are commonplace. In certain situations, there are no type I singularities: Wang [49] showed that there are no type I singularities for almost-calibrated Lagrangian mean curvature flow. We show in Chapter 4 that there are no type I singularities for monotone Lagrangians in Kähler–Einstein manifolds with $\kappa \neq 0$. Hence it becomes important to understand type II singularities in Lagrangian mean curvature flow.

Here the state of the art are the compactness results of Neves, originally established for zero-Maslov Lagrangians in [33] but later extended to monotone Lagrangians in [34]. We present them in the latter form since it is more applicable to the subject matter of this thesis. Compact monotone Lagrangians in $\mathbb{C}^n$ have a maximal existence time strictly controlled by the monotone constant. One can always normalise by homotheties of the ambient space so that this maximal time of existence is $1/2$. Neves' theorems concern singularities happening before this time.

Theorem 2.3.1 (Neves' Theorem A). *Let L be a normalised monotone Lagrangian in $\mathbb{C}^n$ developing a singularity at $T < 1/2$. For any sequence of rescaled flows L_s^j at the singularity with Lagrangian angles θ_s^j, there exists a finite set of angles $\{\bar{\theta}_1, \dots, \bar{\theta}_N\}$ and special Lagrangian cones $L_1, \dots, L_N$ such that after passing to a subsequence we have that for any smooth test function ϕ with compact support, every $f \in C^2(S^1)$ and $s < 0$*

$$\lim_{j\to\infty} \int_{L_s^j} f(\exp(i\theta_s^j))\phi \, d\mathscr{H}^n = \sum_{k=1}^N m_k f(\exp(i\bar{\theta}_k))\mu_k(\phi),$$

where μ_k and m_k are the Radon measure of the support of L_k and its multiplicity respectively.

Furthermore the set of angles is independent of the sequence of rescalings.

Theorem B applies for monotone Lagrangians in $\mathbb{C}^2$.

Theorem 2.3.2 (Neves' Theorem B). *Let L be a normalised monotone Lagrangian in $\mathbb{C}^2$ developing a singularity at $T < 1/2$. For any sequence of rescaled flows L_s^j at the singularity with Lagrangian angles θ_s^j, and for any sequence of connected components Σ_j of $L_s^j \cap B_{4R}(0)$ intersecting $B_R(0)$, there exists a unique angle $\bar{\theta}$ and special Lagrangian cone Σ such that after passing to a subsequence we have that for any smooth test function ϕ on $B_{2R}(0)$ with compact support, every $f \in C^2(S^1)$ and $s < 0$*

$$\lim_{j\to\infty} \int_{\Sigma_j} f(\exp(i\theta_s^j)\phi \, d\mathscr{H}^n = mf(\exp(i\bar{\theta}))\mu(\phi),$$

where μ and m are the Radon measure of the support of Σ and its multiplicity respectively.

Heuristically, these theorems give the type I blow-up models of type II singularities of Lagrangian mean curvature flow as (unions of) special Lagrangian cones. Consider the $n = 2$ case: by considering the hyper-Kähler rotation, one sees that the only special Lagrangian cones are unions of special Lagrangian planes with equal Lagrangian angle. Assuming all planes are multiplicity 1, there is then only one blow-up model up to rotation, a union of two transversely intersecting special Lagrangian planes with the same Lagrangian angle.

We can further characterise singular behaviour by a procedure called the type II blow-up. The precise details of this procedure are not important to this thesis, but we sketch the general principle. We refer the reader to Mantegazza [31] for additional details. Instead of blowing up at a parabolic rate and at a fixed point in time and space, we blow up at a sequence of space-time points (x_i, t_i) maximising the second fundamental form $|A|$ on the interval $[0, T - 1/i]$, at a rate dictated by the second fundamental form $|A|$. Thus we guarantee convergence locally smoothly to an eternal mean curvature flow, i.e. a mean curvature flow existing for all times $t \in (-\infty, \infty)$ (as opposed to the self-shrinkers found by the type I procedure, which are ancient but not eternal).

Note that the type II blow-up is not unique and doesn't have to satisfy the same asymptotics as the type I blow-up. There are few results on type II blow-ups for Lagrangian mean curvature flow so far, but the most important appears in the work of Wood [51], where he shows that almost-calibrated Lagrangian cylinders with prescribed asymptotic behaviour achieve type II singularities in finite time, and the type II blow-up is given by a special Lagrangian called a Lawlor neck asymptotic to the type I blow-up.

Lawlor necks are best described as the hyperKähler rotation of the complex curves $\{zw = c \neq 0\} \subset \mathbb{C}^2$. It is a fundamental question whether these are the generic singularity model for Lagrangian mean curvature flow.

We note one final property of singularities of mean curvature flow. In general, geometric flows may have infinite-time singularities; indeed, this occurs in Ricci flow and in Yang–Mills flow, amongst others. In mean curvature flow however, one can rule out infinite-time singularities in certain cases. For simplicity, we state the following result of Chen–He [10] as we need it in this paper, though it applies in a far wider generality.

Proposition 2.3.3. *Let L be a Lagrangian mean curvature flow of a compact Lagrangian in a compact Kähler–Einstein manifold M with $\kappa > 0$. Then L either attains a finite-time singularity or has uniformly bounded $|A|^2$ for all time and converges subsequentially to a minimal submanifold in M in infinite time.*

Chapter 3

Lagrangian mean curvature flow of the Clifford torus in $\mathbb{C}^2$

3.1 The self-shrinking Clifford torus

We define the self-shrinking Clifford torus L_{Cl} in two equivalent ways. Firstly, as a product torus,

$$L_{\mathrm{Cl}} = \left\{ \sqrt{2} \left(e^{i\theta_1}, e^{i\theta_2} \right) : \theta_1, \theta_2 \in \mathbb{R} \right\} \subset S^3(2) \subset \mathbb{C}^2.$$

Secondly, as an S^1-equivariant Lagrangian,

$$L_{\mathrm{Cl}} = \left\{ 2e^{i\phi} \left(\cos\alpha, \sin\alpha \right) : \phi, \alpha \in \mathbb{R} \right\} \subset S^3(2) \subset \mathbb{C}^2.$$

These two descriptions are equivalent: to see this, first apply the unitary transformation

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

and then apply the change of coordinates

$$\phi = \frac{\theta_1 + \theta_2}{2} \quad \text{and} \quad \alpha = \frac{\theta_1 - \theta_2}{2}. \tag{3.1.1}$$

The choice of scaling here is necessary to ensure that L_{Cl} satisfies the self-shrinker equation

$$\vec{H} = -X^{\perp}/2, \tag{3.1.2}$$

(here, $X^{\perp}$ denotes the projection of the position vector onto the normal bundle.) Let $X : (\theta_1, \theta_2) \mapsto \sqrt{2}\left(e^{i\theta_1}, e^{i\theta_2}\right)$. Then

$$X^*\Omega = -2e^{i(\theta_1+\theta_2)} \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2.$$

where $\Omega = dz_1 \wedge dz_2$ is the standard holomorphic volume form on $\mathbb{C}^2$. The volume form on L is

$$\mathrm{vol}_L = 2d\theta_1 \wedge d\theta_2, \tag{3.1.3}$$

hence the Lagrangian angle is

$$\theta = \theta_1 + \theta_2 + \pi.$$

From this, it is clear that

$$\vec{H} = J\nabla\theta = -\sqrt{2}\left(e^{i\theta_1}, e^{i\theta_2}\right),$$

which verifies that L_{Cl} is a self-shrinker. Satisfying (3.1.2) implies that the Clifford torus attains a finite-time singularity at the origin, with type I blow-up given by L_{Cl}.

There are two classes of J-holomorphic discs with Maslov index 2 with boundary on L_{Cl}, each occurring in an S^1-family. They are

$$\begin{aligned} u_1^{\psi} &: z \mapsto \sqrt{2}\left(z, e^{i\psi}\right) \\ u_2^{\psi} &: z \mapsto \sqrt{2}\left(e^{i\psi}, z\right), \end{aligned}$$

We denote the former class by $\alpha_1 \in H_2(\mathbb{C}^2, L_{\mathrm{Cl}})$, and the latter class by $\alpha_2 \in H_2(\mathbb{C}^2, L_{\mathrm{Cl}})$. In the equivariant description, the profile curve $\gamma(s) = 2e^{is}$ bounds a Maslov 4 disc in the class $\alpha_1 + \alpha_2$. It is immediately clear from observation that the Clifford torus is monotone.

3.2 Instability of the Clifford torus

In this section, we discuss the first part of Theorem 1.2.1, namely the following result:

Theorem 3.2.1. *Let $L_{Cl} \subset S^3(2) \subset \mathbb{C}^2$ be a self-shrinking Clifford torus. Then there exists a C^k-small Hamiltonian perturbation of L_{Cl} such that the flow achieves a type II singularity.*

As commented above, this result was already known ([18], [34]) for large Hamiltonian perturbations.

We review the method used in [16]. We find by delicate but direct calculation that the F-functional takes a local maximum at the self-shrinking Clifford torus L_{Cl} with respect to a particular Hamiltonian variation. We then upgrade this result to an entropy calculation, showing that L_{Cl} is entropy-unstable and hence any perturbation in this direction cannot converge to a Clifford torus after rescaling.

Consider the equivariant description, where L_{Cl} is given as an S^1-bundle over a profile curve $\gamma(s) \in \mathbb{C}$. It is clear from this description that any Hamiltonian perturbation γ' of γ in $\mathbb{C}$ lifts to give a Hamiltonian perturbation of L_{Cl} as a torus in $\mathbb{C}^2$. This follows since any Hamiltonian perturbation of γ preserves the area of the disc class $\alpha_1 + \alpha_2$ interior to γ, and only disturbs the Maslov 0 class $\alpha_1 - \alpha_2$. Thus the monotonicity is preserved with the same constant.

Thus we consider the Hamiltonian perturbation generated by the one-parameter subgroup A_s of $\mathrm{SL}(2,\mathbb{R})$ given by

$$A_s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}.$$

Explicitly, the perturbation is then

$$L_{A_s} = \{2(e^s \cos\phi + ie^{-s}\sin\phi)(\cos\alpha, \sin\alpha) \,:\, \phi, \alpha \in \mathbb{R}\},$$

or in terms of the product torus

$$L_{A_s} = \{\sqrt{2}(\cosh s e^{i\theta_1} + \sinh s e^{-i\theta_2}, \sinh s e^{-i\theta_1} + \cosh s e^{i\theta_2}) \,:\, \theta_1, \theta_2 \in \mathbb{R}\}.$$

We emphasise that both circles in the product torus (corresponding to the disc classes α_1, α_2) do not change size under this perturbation. Instead, the perturbation is a squashing in a Maslov 0 direction.

We show that L_{A_s} does not converge to the self-shrinking Clifford torus under rescaled mean curvature flow. We first recall the definitions and basic properties of the F-functional and the entropy λ. We follow [12].

For a compact immersion $X : L \to \mathbb{C}^2$, we define the F-functional

$$F(X, x_0, t_0) = \frac{1}{4\pi t_0} \int_L \exp\left(-\frac{|X - x_0|^2}{4t_0}\right) \mathrm{vol}_L,$$

and the entropy as

$$\lambda(X) = \sup_{(x_0, t_0) \in \mathbb{C}^2 \times \mathbb{R}^+} F(X, x_0, t_0).$$

The entropy has the following properties:

Lemma 3.2.2.

(a) The entropy is invariant under translations, dilations and rotations.

(b) The entropy is non-increasing under mean curvature flow and rescaled mean curvature flow.

(c) The critical points of the entropy are (after potential translation and dilation) self-shrinkers.

Here rescaled mean curvature flow is the standard parabolic rescaling about any space-time point.

For a self-shrinker X, we have that

$$\lambda(X) = F(X, 0, 1).$$

It is therefore straightforward to compute the entropy of the Clifford torus.

Lemma 3.2.3. *For the Clifford torus $X : L \to \mathbb{C}^2$, we have*

$$\lambda(X) = \frac{2\pi}{e} = 2.311\ldots$$

Proof. We compute

$$\lambda(X) = \frac{1}{4\pi}\int_L \exp\left(-\frac{1}{4}|X|^2\right) \mathrm{vol}_L = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} 2e^{-1}\, d\theta_1\, d\theta_2 = \frac{4\pi^2}{2\pi e} = \frac{2\pi}{e},$$

where we used $|X|^2 = 4$ and (3.1.3). $\square$

We now proceed by estimating the value of the F-functional along the variation A_s described above. The calculations in this section can be found in greater detail in [16], though significant portions of them were performed in Mathematica.

Proposition 3.2.4. *For s near 0, we have that*

$$F(X(s),0,1) - F(X(0),0,1) = -\frac{4\pi}{9e}s^6 + O(s^8).$$

Hence $F(X(s),0,1)$ has a strict local maximum at $s = 0$.

Proof. Recall that

$$X(s) = \sqrt{2}(\cosh s e^{i\theta_1} + \sinh s e^{-i\theta_2}, \sinh s e^{-i\theta_1} + \cosh s e^{i\theta_2}). \tag{3.2.1}$$

It follows by direct calculation that

$$F(X(s),0,1) = \frac{1}{2\pi e}\int_0^{2\pi}\int_0^{2\pi} I(s)\, d\theta_1\, d\theta_2,$$

where

$$I(s) = \sqrt{\cosh^2 2s - \sinh^2 2s \cos^2(\theta_1+\theta_2)}\, e^{1-\cosh 2s - \sinh 2s \cos(\theta_1+\theta_2)}.$$

We notice that $I(s)$ is a real analytic function of s. Therefore, we seek a power series expansion about $s = 0$. We have that $I(0) = 1$, which verifies Lemma 3.2.3. Further calculation reveals that the integral of the k-th derivative $I^{(k)}(0)$ is equal to 0 for $k \leq 5$, and that

$$\int_0^{2\pi}\int_0^{2\pi} \frac{I^{(6)}(0)}{6!}\, d\theta_1\, d\theta_2 = -\frac{8}{9}\pi^2.$$

The integral of $I^{(7)}(0)$ is also 0, so the result follows. □

We now consider the value of the F-functional for L_{A_s} for space-time centres near $(0,1)$.

Proposition 3.2.5. *Let $X(s)$ denote the position of L_{A_s} as in* (3.2.1). *Then there exists $s_0 > 0$ and $r_0 > 0$ such that whenever (x_0,t_0) lies in the set*

$$S = \{(x_0,t_0) \in \mathbb{C}^2 \times \mathbb{R}^+ : |x_0|^2 + 2|t_0 - 1|^2 \leq r_0^2\},$$

and $|s| \leq s_0$ we have

$$F(X(s),x_0,t_0) \leq F(X(0),0,1) - \frac{\pi}{4e}(|x_0|^2 + 2|t_0-1|^2) - \frac{2\pi}{9e}s^6.$$

Proof. The proof is similar to that of Proposition 3.2.4. This time, we have

$$F(X(s),x_0,t_0) = \frac{1}{2\pi e}\int_0^{2\pi}\int_0^{2\pi} I(s,x_0,t_0)\, d\theta_1\, d\theta_2,$$

for $x_0 \in \mathbb{C}^2$ and $t_0 \in \mathbb{R}^+$, where

$$I(s,x_0,t_0) = \frac{1}{t_0}\sqrt{\cosh^2 2s - \sinh^2 2s \cos^2(\theta_1+\theta_2)}\, e^{1 - \frac{|X(s)-x_0|^2}{4t_0}}.$$

Pick any $(\xi,\tau) \in \mathbb{C}^2 \times \mathbb{R}$ with $|\xi|^2 + 2|\tau|^2 = 1$, and define

$$f(r,s) = \int_0^{2\pi}\int_0^{2\pi} I(s, r\xi, 1 + r\tau)\, d\theta_1\, d\theta_2.$$

Performing a Taylor expansion using (3.2.1) (and with the help of Mathematica) around $(r,s) = (0,0)$ yields

$$f(r,s) = f(0,0) - \pi^2 r^2 - \frac{8}{9}\pi^2 s^6 + O(r^2 s) + O(r^3) + O(s^7).$$

We can thus choose $r_0, s_0 > 0$ sufficiently small such that for $|r| \leq r_0$ and $|s| \leq s_0$ we have

$$f(r,s) \leq f(0,0) - \frac{1}{2}\pi^2 r^2 - \frac{4}{9}\pi^2 s^6.$$

Since this estimate is uniform in (ξ, τ), this yields the desired statement. □

We can now combine Propositions 3.2.4 and 3.2.5 to give our first key result.

Theorem 3.2.6. *For s near 0 we have that*

$$\lambda(X(s)) \leq \lambda(X(0)) - \frac{2\pi}{9e}s^6.$$

Hence, the entropy $\lambda(X(s))$ has a local maximum at $s = 0$.

Proof. Recall that Huisken's monotonicity formula [25] implies that for a compact self-shrinker L, the entropy $\lambda(L)$ is uniquely attained at $(0,1)$: Since the flow is self-similar, we have that

$$\lambda(L) = F(L, 0, 1).$$

Now assume that there is a point $(x_0, t_0) \neq (0,1)$ such that $\lambda(L) = F(L, x_0, t_0)$. The monotonicity formula then implies that L is also a self-shrinker with respect to the point $(x_0, t_0 - 1)$. So $t_0 = 1$. Furthermore, the monotonicity formula implies that the entropy is attained on any point along the line containing x_0 and 0, and thus L is a product $L' \times \mathbb{R}$. This contradicts the compactness of L, so the entropy is attained uniquely at $(0,1)$.

To apply Proposition 3.2.5, it remains to show that the entropy of $X(s)$ is attained in the set S for sufficiently small $|s| < s_0$. We can choose s_0 such that $X(s)$ has arbitrarily small C^1-norm as an exponential graph over $X(0)$.

Now consider L' given as an exponential normal graph of $U \in C^\infty(NL)$. We choose $\varepsilon_0 > 0$ and assume

$$\|U\|_{C^1} \leq \varepsilon \leq \varepsilon_0. \tag{3.2.2}$$

Note that this implies that for $\varepsilon_0 = \varepsilon_0(L) > 0$ sufficiently small, given any $\eta_0 > 0$, there exists a $\delta_0 = \delta_0(L, \eta_0) > 0$ such that

$$F(L', x_0, t_0) \leq 1 + \eta_0 \tag{3.2.3}$$

for all $x_0 \in \mathbb{C}^2$, $0 < t_0 < \delta_0$. We can choose $\eta_0 = \frac{1}{4}(\lambda(L) - 1) > 0$ (as L is not a plane). Since the entropy of L is uniquely attained at $(0,1)$, given any $r > 0$, there exists $0 < \eta < \eta_0$ such that

$$F(L, x_0, t_0) < \lambda(L) - 3\eta$$

for all $|x_0| > r$ and $(t_0 - 1)^2 > r$. Using (3.2.3) we see that we can thus choose ε sufficiently small in (3.2.2) such that

$$F(L', x_0, t_0) < \lambda(L) - 2\eta$$

for all $|x_0| \geq r$ and $(t_0 - 1)^2 \geq r$ and

$$F(L', 0, 1) \geq \lambda(L) - \eta.$$

We deduce that the entropy of L' is attained in the set

$$\{(x_0, t_0) \in \mathbb{C}^2 \times \mathbb{R}^+ : |x_0| \leq r, (t_0 - 1)^2 \leq r\}.$$

Applying this to our set-up, we see that for s small, the entropy $\lambda(X(s))$ is only attained at (possibly non-unique) points (x_s, t_s) with the property $(x_s, t_s) \to (0, 1)$ as $s \to 0$. The claimed result then follows directly from Proposition 3.2.5. □

Theorem 3.2.7. *For any $s > 0$, the torus $X(s)$ does not converge to the self-shrinking Clifford torus after rescaling.*

Proof. Note the symmetry of $X(s)$ implies there is only one possible self-shrinking Clifford torus to converge to after rescaling, namely $X(0)$. Entropy is non-increasing under (rescaled) mean curvature flow and the entropy $\lambda(X(s))$ is strictly less than $\lambda(X(0))$ for any $s > 0$, so we see $X(s)$ cannot converge to $X(0)$ after rescaling. □

3.3 Stability of the Clifford torus

In contrast to the previous section, we find the following:

Theorem 3.3.1. *Any Hamiltonian deformation of L_{Cl} restricted to the 3-sphere $S^3(2)$ forms a type I singularity at the origin with type I blow-up given by L_{Cl}.*

We call Lagrangians in $\mathbb{C}^n$ lying in the $(2n-1)$-sphere of radius r *spherical*.

The proof relies on a result relating Lagrangian mean curvature flow of spherical Lagrangian tori to Lagrangian mean curvature flow of Lagrangian tori in complex projective space. The result was first explored by Castro–Lerma–Miquel [9].

Proposition 3.3.2. *Let $n > 1$. A spherical Lagrangian torus $L = T^n \subset \mathbb{C}^n$ is normalised monotone if and only if $L/S^1 = T^{n-1}$ is a monotone Lagrangian torus in $\mathbb{CP}^{n-1}$, where the quotient is the Hopf fibration $S^{2n-1}/S^1 = \mathbb{CP}^{n-1}$.*

Proof. Suppose L is spherical Lagrangian. Then the normal vector field N to $S^{2n-1} \subset \mathbb{C}^n$ is normal to L at every point, so JN is tangent to L. The integral curves of JN are Hopf circles by definition, so L is foliated by Hopf circles. Hence the quotient L/S^1 is well-defined.

The Hopf fibration $f : S^{2n-1} \to \mathbb{CP}^{n-1}$ induces an isomorphism between the relative homology groups

$$\tilde{f} : H_k(S^{2n-1}, L) \to H_k(\mathbb{CP}^{n-1}, L/S^1)$$

for all k. Furthermore, for $n > 1$, the long exact sequence of relative homology groups for the inclusions $L \subset S^{2n-1} \subset \mathbb{C}^n$ gives an isomorphism between $H_2(S^{2n-1}, L)$ and $H_2(\mathbb{C}^n, L)$. So any relative disc class on L in $\mathbb{C}^n$ can be represented by a disc class in S^{2n-1}.

The Fubini–Study form $\omega_{\mathbb{CP}^{n-1}}$ on $\mathbb{CP}^{n-1}$ is induced from the standard symplectic form ω on $\mathbb{C}^n$ by the Hopf fibration, so up to choosing the correct scaling on S^{2n-1}, $\tilde{f}$ preserves the monotone constant. □

Lemma 3.3.3. *The condition that a Lagrangian is spherical is preserved under mean curvature flow. If $F_0(L)$ has radius $R_0 = |F_0(L)|$, then $F_t(L)$ has radius $R_t = |F_t(L)| = \sqrt{R_0^2 - 2nt}$.*

Proof. Recall that for mean curvature flow of k-dimensional submanifolds of $\mathbb{R}^m$, spheres of radius $R(t) = \sqrt{R^2 - 2kt}$ are barriers on the inside and outside. We claim this implies spherical Lagrangians with initial radius R_0 remain spherical with radius $\sqrt{R_0^2 - 2nt}$. Indeed, suppose this is not the case. Then we can find some time $t' > 0$, $\varepsilon \neq 0$ and point $p \in L$ such that the radius at (p,t') is $\sqrt{R_0^2 - 2nt'} + \varepsilon$. Without loss of generality, assume $\varepsilon > 0$. Then the sphere of radius $R(t) = \sqrt{R_0^2 + \varepsilon^2/4 - 2nt}$ does not intersect $F_t(L)$ for all time, yet at $t = t'$, $R(t') < \sqrt{R_0^2 - 2nt'} + \varepsilon$, a contradiction. □

Let $F_0 : L \to \mathbb{C}^n$ be a spherical Lagrangian with radius R_0. Then $F_t(L)$ is spherical for all time by Lemma 3.3.3. Thus we can write $F_t(L) = \frac{1}{\sqrt{R_0^2 - 2nt}} \tilde{F}_t(L)$ for embeddings $\tilde{F}_t : L \to S^{2n-1}(1)$ of L into the unit sphere. Define $G_{\tau(t)}(L/S^1) = f(\tilde{F}_t(L))$ for some function $\tau(t)$ to be determined.

We claim that we can choose $\tau(t)$ such that the flow G_τ is a mean curvature flow in $\mathbb{CP}^{n-1}$. As in [9], we observe the following:

1. The mean curvature vector $\vec{H}$ of $F_t(L) \subset \mathbb{C}^n$ is related to the mean curvature vector $\bar{H}$ of $\tilde{F}_t(L) \subset S^{2n-1}(1)$ by
$$\vec{H} = \frac{1}{\sqrt{R_0^2 - 2nt}} \bar{H} - \frac{2}{\sqrt{R_0^2 - 2nt}} F_t.$$
2. Since f is a Riemannian submersion, the mean curvature vector $\tilde{H}$ of $\tilde{F}_t(L) \subset S^{2n-1}(1)$ is related to the mean curvature vector $\bar{H}$ of $f(\tilde{F}_t(L)) \subset \mathbb{CP}^{n-1}$ by $\tilde{H} = \bar{H}^*$ where $\bar{H}^*$ is the horizontal lift of $\bar{H}$ under f.

Therefore, choosing
$$\tau(t) = \frac{1}{2n} \log \left(\frac{R_0^2 - 2nt}{R_0^2} \right)$$
implies that $G_\tau : L/S^1 \to \mathbb{CP}^{n-1}$ is a mean curvature flow.

We summarise the above as follows:

Proposition 3.3.4. *Mean curvature flow of spherical monotone Lagrangians in $\mathbb{C}^n$ induces a mean curvature flow of Lagrangians in $\mathbb{CP}^{n-1}$ by the Hopf fibration, and the converse also holds.*

Suppose we have a spherical Lagrangian $F_0 : L \to \mathbb{C}^n$ with radius R_0 as the initial condition. Then singularities of $F_t(L)$ before time $R_0^2/2n$ correspond to finite-time singularities of $G_\tau(L/S^1)$.

If $G_\tau(L/S^1)$ converges as varifolds to a minimal Lagrangian in $\mathbb{CP}^{n-1}$ in infinite time, then $\tilde{F}_t(L)$ converges as varifolds to a Lagrangian submanifold of $\mathbb{C}^n$, minimal as a submanifold of $S^{2n-1}(1)$. Furthermore, the rescaling $\tilde{F}_t$ is equivalent to the standard parabolic rescaling, so the same holds for the parabolic rescaling.

Returning to the case of Clifford tori in $\mathbb{C}^2$, we see that Theorem 3.3.1 follows directly from Proposition 3.3.4 and the Proposition 2.2.7.

Chapter 4

Lagrangian mean curvature flow in the complex projective plane

4.1 Lagrangian mean curvature flow in Fano manifolds

We now switch attention to Lagrangian mean curvature flow in Kähler–Einstein manifolds with positive Einstein constant $\kappa > 0$. We call these *Fano manifolds* (though we remark that this definition of Fano manifolds is somewhat non-standard in the literature), and the fundamental example is complex projective space $\mathbb{CP}^n$ with the Fubini–Study metric which has $\kappa = 2(n+1)$.

Recall Proposition 2.2.7 on curve-shortening flow on the sphere. We saw that monotone curves did not attain type I singularities: heuristically, any type I singularity would require the collapsing of one of the disc classes, which is prohibited by the monotone condition. We now generalise this to higher dimensions. First, we can classify all zero-Maslov self-shrinkers that may arise as a type I blow-up by a result of Groh–Schwarz–Smoczyk–Zehmisch [18]:

Theorem 4.1.1. *If $F : L^n \to \mathbb{C}^n$ is a zero-Maslov Lagrangian self-shrinker arising as a result of a type I blow-up, then L is a minimal Lagrangian cone.*

This follows directly from [18, Theorem 1.9], noting that type I blow-ups have bounded area ratios.

Since type I blow-ups are smooth, embedded self-shrinkers for type I singularities, this implies there are no zero-Maslov type I blow-ups for type I singularities. Since any type I model is locally symplectomorphic to the standard unit ball, this excludes the possibility of type I singularities for monotone Lagrangians:

Theorem 4.1.2. *Let $F_t : L^n \to M^{2n}$ be a monotone Lagrangian mean curvature flow, $\kappa \neq 0$. Then F_t does not attain any type I singularities.*

Proof. Suppose for a contradiction that F_t attains a type I singularity at time T. Any sequence $\eta_i \to \infty$ subsequentially defines a type I blow-up

$$\tilde{F}_s := \lim_{i\to\infty} \tilde{F}_s^{\eta_i} = \lim_{i\to\infty} \eta_i F_{T+\eta_i^{-2}s}.$$

Since the singularity is type I, $\tilde{F}(L) := \tilde{F}_{-1}(L)$ is a non-planar embedded Lagrangian self-shrinker, and hence by Theorem 4.1.1 has non-zero Maslov class.

Let $\tilde{D} \in H_2(\mathbb{C}^n, \tilde{F}(L))$ have $\mu(\tilde{D}) > 0$. The convergence of $\tilde{F}_s$ to a type I blow-up is smooth and the Maslov class is topological, so for all sufficiently large i, there exists $\tilde{D}_i \in H_2(\mathbb{C}^n, \tilde{F}_{-1}^{\eta_i})$ with $\mu(\tilde{D}_i) = \mu(\tilde{D}) > 0$ and $\tilde{D}_i \to \tilde{D}$ as $i \to \infty$. Furthermore, $\tilde{D}^i$ are the images under the parabolic rescaling of discs $D_i = \eta_i^{-1}\tilde{D}_i \in \pi_2(W, F_{T-\eta_i^{-2}}(L))$. Since L is monotone and the Maslov class is invariant under rescaling,

$$\int_{D_i} \omega = \frac{\pi}{\kappa}\mu(\tilde{D}_i) = \frac{\pi}{\kappa}\mu(\tilde{D}) > 0$$

for all i, but

$$\lim_{i\to\infty} \int_{D_i} \omega = \lim_{i\to\infty} \int_{\eta_i^{-1}\tilde{D}_i} \omega = 0,$$

a contradiction. □

This theorem is the positive curvature equivalent of the result of Wang [49] showing that almost-calibrated Lagrangians do not attain type I singularities in Calabi–Yau manifolds. This strengthens the perspective that monotone submanifolds are the correct class of submanifolds to study to find positive curvature analogues of the Thomas–Yau conjecture. The rest of the thesis will be devoted to exploring what a Thomas–Yau conjecture looks like in the prototypical Fano surface $\mathbb{CP}^2$.

4.2 Lagrangian tori in $\mathbb{CP}^2$

We study Lagrangian mean curvature flow of tori in $\mathbb{CP}^2$ with the Fubini–Study metric. Recall that as in Example 2.1.3, this is a Kähler–Einstein manifold with Einstein constant $\kappa = 6$. The Kähler form induced by the Hopf fibration is unique up to symplectomorphisms.

The Clifford torus

$$L_{\mathrm{Cl}} = \{[x:y:z] : |x| = |y| = |z| = 1\} \subset \mathbb{CP}^2$$

is a Lagrangian submanifold. It is minimal and monotone. The relative homology class $H_2(\mathbb{CP}^2, L_{\mathrm{Cl}})$ is generated by 3 Maslov 2 J-holomorphic discs:

$$\begin{aligned}
u_0 &: w \mapsto [w:1:1] \\
u_1 &: w \mapsto [1:w:1] \\
u_2 &: w \mapsto [1:1:w].
\end{aligned}$$

where $w \in D$, the unit disc.

A natural question to ask is whether the Clifford torus is the unique monotone Lagrangian torus in $\mathbb{CP}^2$ up to Hamiltonian isotopy. The answer is definitively no.

Chekanov–Schlenk [40] prove the existence of a monotone torus called the Chekanov torus L_{Ch} which is Lagrangian-isotopic to L_{Cl} but not Hamiltonian-isotopic. Until more recently, the question of whether there were other monotone tori was open, but in a pair of papers Vianna ([47], [48]) showed the existence of a countably infinite family of exotic monotone Lagrangian tori, with no two Hamiltonian isotopic. We review Vianna's construction, which relies on the machinery of almost-toric manifolds, in the sequel.

4.2.1 Almost-toric manifolds

Definition 4.2.1. Let (M^{2n}, ω) be a compact symplectic manifold, and let G act on M by an effective Lie group action. The action is called *Hamiltonian* if the vector field X_ξ generated by an element ξ of the Lie algebra $\mathfrak{g}$ is given by the symplectic gradient X_{f_ξ}

of a smooth function f_ξ on M. [1]

A *moment map* for a Hamiltonian action is a map $\mu : M \to \mathfrak{g}^*$ defined by

$$\mu(x)(\xi) = f_\xi$$

In the case that $G = T^n$, M is called a *toric manifold*, and the projection $\mu : M \to \mathbb{R}^n$ is called a *toric fibration*, since the pre-image of each point is a T^m-torus with $0 \le m \le n$.

The following theorem is fundamental to toric geometry. The first result is due to Atiyah [3] and Guillemin–Sternberg [19], and the second due to Delzant [13].

Theorem 4.2.2.

1. *Let M be a compact toric manifold. Then the image $\mu(M)$ of μ is a convex polytope in $\mathbb{R}^n$.*
2. *The image of the moment map determines M, the symplectic structure and the action.*

Note that ω vanishes on the fibre above any point p of a toric fibration, and hence the fibre is Lagrangian when it is n-dimensional.

By way of example, and also since it is the object of study for the rest of the thesis, we consider the case of $\mathbb{CP}^2$ in more detail. Let $[x : y : z]$ be homogeneous coordinates on $\mathbb{CP}^2$ and consider the moment map

$$\mu([x : y : z]) = \frac{1}{|x|^2 + |y|^2 + |z|^2}\left(|x|^2, |y|^2\right). \tag{4.2.1}$$

The image $\mu(\mathbb{CP}^2)$ is a triangular polytope with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. The fibre above any interior point of the triangle is a Lagrangian torus. Of particular note is the fibre above the barycentre $(1/3, 1/3)$, which is a monotone, minimal Clifford torus.

The fibration is singular at the edges and vertices; the fibre over an edge point is a circle, and the fibre over a vertex is a single point. The pre-image of each edge of the polytope are holomorphic spheres in $\mathbb{CP}^2$ (the "lines at infinity").

[1] Recall that the symplectic gradient of f is defined to be the vector field X_f satisfying $\omega(X_f, \cdot) = df$, although here sign convention differs amongst authors.

Symington–Leung [29] define an extension of toric manifolds to include a wider range of singular points. Restricting our attention to 4-manifolds for clarity, let $f : M^4 \to B^2$ be a fibration of a symplectic manifold. Then M is called *almost-toric* if for any point p in M there exists a Darboux neighbourhood U of p with local coordinates $(z_j) = (x_j + iy_j)$ on U, $\omega = dx \wedge dy$, such that in those coordinates $f|_U$ is given by 1 of 4 possibilities:

$$f|_U(z_1,z_2) = \begin{cases} (x_1,x_2) & \text{regular point} \\ (x_1,x_2^2+y_2^2) & \text{elliptic, cylindrical singularity} \\ (x_1^2+y_1^2,x_2^2+y_2^2) & \text{elliptic, toric singularity} \\ (x_1y_1+x_2y_2,x_1y_2-y_1x_2) & \text{nodal singularity} \end{cases} \tag{4.2.2}$$

Remark 4.2.3.

1. The first two singularity models occur in toric fibrations, at the edges and vertices of the base.

2. The above definition is motivated by a result from Eliasson [14], extending work of Williamson [50], which shows that these are precisely the singularities that occur in integrable Hamiltonian systems.

   Symmington–Leung [29] consider one additional type of singularity which appears in integrable Hamiltonian systems, called hyperbolic singularities. We ignore these singularities for a few reasons. Firstly, they impose a different structure on the base to nodal and elliptic singularities, causing the base to become non-smoothable. Secondly, they do not seem likely as a singularity model for Lagrangian mean curvature flow in reasonable situations.

3. The 3 singularity models in (4.2.2) are all found in Lagrangian mean curvature flow. The first two are the self-shrinking Lagrangian cylinder and Clifford torus respectively, and the last is the Lawlor neck singularity observed by Neves [33] and constructed explicitly by Wood [51].

4. One of the key features of this definition is that the moment map has been forgot-

ten, leaving instead the purely topological consideration of the Lagrangian fibration.

4.2.2 Vianna's exotic tori in $\mathbb{CP}^2$

Vianna constructs in [48] an infinite family $\mathscr{F}$ of monotone Lagrangian tori in $\mathbb{CP}^2$, with no two tori Hamiltonian isotopic. We present some details of this construction here as it forms the main motivating example for the rest of the thesis.

The first member of the family $\mathscr{F}$ is the Clifford torus

$$L_{\mathrm{Cl}} = \{[x:y:z] : |x| = |y| = |z| = 1\},$$

which is realised as the barycentric fibre in the toric fibration given by μ in (4.2.1). From here, Vianna constructs the next member of $\mathscr{F}$ by a topological procedure known as a *mutation*:

1. Introduce a nodal fibre at one of the corners by a *nodal trade*. The corner of the base diagram is now a circle and the fibre above the cross is a Lagrangian torus pinched to create a nodal singularity. The barycentric fibre is still a Clifford torus, and the metric is still the Fubini–Study metric.

2. Rescale a neighbourhood of the line at ∞ (i.e. the $\mathbb{CP}^1$ given by $\{[x:y:0] : x,y \in \mathbb{C}\}$) until the barycentre has passed over the nodal fibre. The barycentre is now a Chekanov torus and the metric is no longer the Fubini–Study metric.

3. Isotope the metric back to the Fubini–Study metric using Moser's trick. The barycentre remains a Chekanov torus.

Items 2 and 3 together are called a *nodal slide*, and the full mutation is illustrated in Figure 4.1.

Vianna then iterates this procedure, introducing new nodal fibres at different corners of the moment polytope. Since the two corners $(1,0)$ and $(0,1)$ are the same after the first mutation, the Chekanov torus L_{Ch} becomes a unique new torus $L_{(1,4,25)}$. Iterating further from this point generates two new tori every time. Vianna indexes this family by

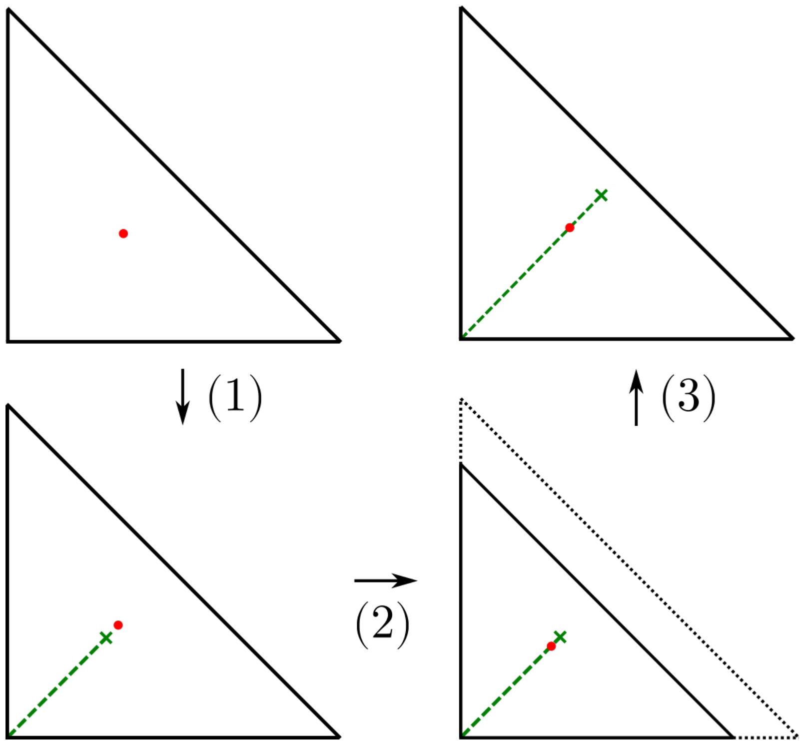

Figure 4.1: The mutation procedure. The fibre above the barycentres (red dots) are Clifford tori on the left-hand side, and Chekanov tori on the right-hand side.

$$\begin{tikzcd}[column sep=small]
& & & & \cdots \\
& & & L_{(1,25,169)} \arrow[ur] \arrow[dr] & \\
& & & & \cdots \\
L_{\mathrm{Cl}} \arrow[r] & L_{\mathrm{Ch}} \arrow[r] & L_{(1,4,25)} \arrow[uur] \arrow[ddr] & & \\
& & & & \cdots \\
& & & L_{(4,25,841)} \arrow[ur] \arrow[dr] & \\
& & & & \cdots
\end{tikzcd}$$

Figure 4.2: Vianna's exotic tori, indexed by Markov triples (a^2,b^2,c^2).

integer triples (a^2,b^2,c^2) with $a^2+b^2+c^2=3abc$, which are known as *Markov triples*, and shows that a torus $L_{(a^2,b^2,c^2)}$ is realised as the barycentric fibre of a degeneration of the weighted projective space $\mathbb{CP}^2(a^2,b^2,c^2)$, though we shall not use this perspective in this thesis. We focus for the rest of the thesis only on the first level of this procedure, which has been known since the work of Chekanov–Schlenk [40].

We can distinguish tori of Clifford-type L_{Cl} from tori of Chekanov-type L_{Ch} by counting J-holomorphic disc classes in $H_2(\mathbb{CP}^2,L)$. Following the results of Auroux [4], we have that there are 3 classes of Maslov 2 discs with boundary on L_{Cl}. Denote by α_1 the disc class $w \mapsto [w:1:1]$, and by α_2 the disc class $w \mapsto [1:w:1]$. Then $H_2(\mathbb{CP}^2, L_{\mathrm{Cl}})$ is generated by $\{\alpha_1, \alpha_2, Q - \alpha_1 - \alpha_2\}$ where $Q = [\mathbb{CP}^1]$ is the hyperplane class.

On the other hand, consider the Chekanov torus in $\mathbb{C}^2 = \mathbb{CP}^2 - \{z = 0\}$ given by

$$L_{\mathrm{Ch}} = \left\{ \left(\gamma(s) e^{i\alpha}, \gamma(s) e^{-i\alpha} \right) : s, \alpha \in \mathbb{R} \right\},$$

where $\gamma(s) \in \mathbb{C}$ is a closed curve not enclosing the origin. By an abuse of notation, let α denote the disc class of a disc with boundary given by the α coordinate, and by β the disc class given by the s coordinate. Then $H_2(\mathbb{CP}^2, L_{\mathrm{Ch}})$ is generated by $\{\alpha, \beta, Q\}$. However, the class α does not contain any holomorphic representatives - this is precisely the same reason the corresponding class can collapse for the Clifford torus L_{Cl} in $\mathbb{C}^2$ as demonstrated in Chapter 3. In fact, the Maslov 2 classes on L_{Ch} are precisely

$$\{\beta, Q - 2\beta + \alpha, Q - 2\beta, Q - 2\beta - \alpha\},$$

each occurring with moduli space of holomorphic discs of dimension 1 except for the $Q - 2\beta$ class which has dimension 2.

We adopt the following terminology for tori in $\mathbb{CP}^2$, which we will attempt to follow whenever there could be confusion for the rest of the thesis:

1. We call the torus $L^0_{\mathrm{Cl}} = \left\{ [x:y:z] : |x|^2 = |y|^2 = |z|^2 \right\}$, and any rotation of it by an element $A \in \mathrm{PU}(3)$, the holomorphic isometry group of $\mathbb{CP}^2$, a minimal Clifford torus.

2. We call any torus in $\mathbb{CP}^2$ a *Clifford-type torus* if it has the same J-holomorphic structure as L^0_{Cl}, i.e. if there is a Lagrangian isotopy to the flat Clifford torus inducing an isomorphism on the relative homology.

3. We call any monotone Clifford-type torus a *Clifford torus* and denote it by L_{Cl}.

We extend this convention in the natural way to Chekanov tori (though we don't know if there are any minimal exotic tori.)

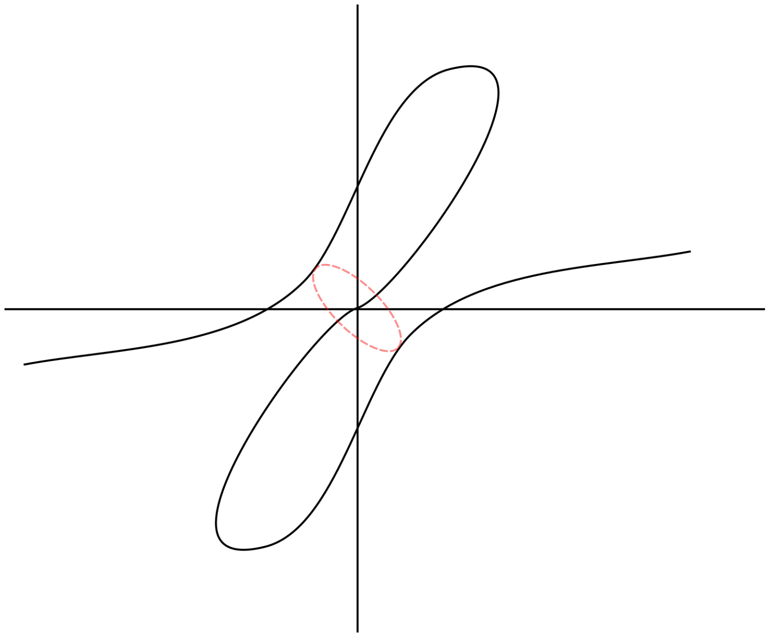

Figure 4.3: An example of a zero-object singularity - A Lagrangian plane attaining a type II singularity.

4.3 Clifford and Chekanov tori in Lefschetz fibrations

Our goal is to study the behaviour of Clifford and Chekanov tori in $\mathbb{CP}^2$ under mean curvature flow, but this presents a number of difficulties. The main problem is the class of potential singularities is too great. Heuristically, singular behaviour is local and since $\mathbb{CP}^2$ looks flat on sufficiently small scales, we expect that a priori any singular behaviour observed for zero-Maslov Lagrangians in $\mathbb{C}^2$ should also occur for monotone Lagrangians in $\mathbb{CP}^2$. In particular, zero-object singularities[2] like those studied in Neves [33, Figure 3] (Figure 4.3) can occur and currently we have little understanding about the nature of these singularities. A second issue is that there is no control over where the singularity happens and what Lagrangian cone the type I blow-up produces, even under the assumption that we obtain Lawlor neck singularities. Since these are general problems in Lagrangian mean curvature flow, we choose a symmetric subclass of Lagrangians in $\mathbb{CP}^2$ which cannot have the zero-object singularities and where we have strong control over the location and type of the singularities.

[2]As a note on the terminology: The obvious surgery at such a singularity bubbles off an immersed Lagrangian sphere with a single transverse self-intersection. Since such a sphere represents a zero object in the Fukaya category, it seems sensible to call these singularities which are collapsing zero-homotopic curves zero-object singularities. See Joyce [27, Section 3.7] for a more detailed description.

We consider two rational maps $\mathbb{CP}^2 \to \mathbb{CP}^1$. The first is the Lefschetz fibration

$$f([x:y:z]) = [xy:z^2]$$

in the complement of the anti-canonical divisor $D = \{(xy - z^2)z = 0\}$. The second is the projection

$$\pi([x:y:z]) = [y:z].$$

This extends to a foliation of $\mathbb{CP}^2$ by holomorphic spheres each intersecting at a single point $[0:0:1]$ with intersection number 1.

We call a subset $U \subset \mathbb{C}$ point-symmetric if $x \in U$ if and only if $-x \in U$. For a point-symmetric curve $\gamma(s) \in \mathbb{C}$, define

$$L^0_\gamma = \left\{ \left[\gamma(s)e^{i\alpha} : \gamma(s)e^{-i\alpha} : 1\right] \,\middle|\, \alpha \in \mathbb{R}, s \in \mathbb{R} \right\}$$

and notice that since γ is point-symmetric, $f(L^0_\gamma) = \{[\gamma(s)^2 : 1] : s \in \mathbb{R}\}$ is an embedded curve in $\mathbb{CP}^1$ if $\gamma(s)$ is embedded in $\mathbb{C}$. We will also allow unions of two smooth non-intersecting curves such that the union is point-symmetric. By an abuse of notation, we refer to such a curve as $\gamma(s)$ where the parameter s is now allowed to vary over two intervals or circles.

First, we identify various Lagrangians in this format. Let $\gamma(s) = s \in \mathbb{R} \subset \mathbb{C}$. Then

$$\{[se^{i\alpha} : se^{-i\alpha} : 1]\}$$

lies above γ and is compactified by the circle $[e^{i\alpha} : e^{-i\alpha} : 0]$ at infinity. The resulting manifold is

$$L^0_\mathbb{R} := \{[se^{i\alpha} : se^{-i\alpha} : 1]\} \cup \{[e^{i\alpha} : e^{-i\alpha} : 0]\} = \{[1 : e^{-2i\alpha} : re^{-i\alpha}]\} \cup \{[0:0:1]\}$$

or equivalently, using the substitution $s = \cot\phi$,

$$L^0_\mathbb{R} = \{[\cos\phi e^{i\alpha} : \cos\phi e^{-i\alpha} : \sin\phi] : \phi \in [0, \pi/2], \alpha \in \mathbb{R}\}.$$

Note that $L^0_{\mathbb{R}}$ is fixed under the anti-symplectic involution $X : [x : y : z] \mapsto [\bar{y} : \bar{x} : \bar{z}]$, hence is isomorphic to $\mathbb{RP}^2$. The same applies for any other line through the origin in $\mathbb{C}$.

The curve $\gamma_r(s) = re^{is}$ lifts to a Lagrangian torus of Clifford-type, which is monotone and minimal if and only if $r = 1$. Furthermore, any point-symmetric closed curve enclosing the origin lifts to a torus of Clifford-type, monotone if and only if the symplectic area contained is equal to $4\pi/6 = 2\pi/3$. This follows from the Cieliebak–Goldstein formula (2.2.3), $\kappa = 6$ and the fact that the disc is Maslov 4. Any closed circle γ not enclosing the origin and its point-symmetric image $-\gamma$ together lift to a torus of Chekanov-type (provided γ does not intersect $-\gamma$), monotone if and only if the area contained is $2\pi/6 = \pi/3$. The fact that these Lagrangians are Clifford and Chekanov respectively can be checked by observing their images under the Lefschetz fibration f and comparing with the standard definitions in Auroux, for instance, [4].

We will distinguish between Clifford tori and Chekanov tori by their intersections with real projective planes $\mathbb{RP}^2$. Immediately we observe that any closed curve γ enclosing the origin intersects any line l through the origin in at least two points, hence any equivariant Clifford torus $L_\gamma \cong L_{\mathrm{Cl}}$ intersects $L_l \cong \mathbb{RP}^2$ in at least one circle. Indeed, this result is generalisable: L_{Cl} is non-displaceable from $\mathbb{RP}^2$, as can be shown in multiple different ways (see for instance [5] or [15]). Indeed, Amorim and Alston [1] give a lower bound of 2 for the number of intersections between a Clifford torus and $\mathbb{RP}^2$. On the other hand, one can easily observe that there exists a pair of point-symmetric circles $\gamma(s) \in \mathbb{C}$ each containing a disc of area 2 and not intersecting the imaginary axis $i\mathbb{R} \in \mathbb{C}$. Hence Chekanov tori are displaceable from $\mathbb{RP}^2$.

In the sequel, it will be useful to consider cones of real projective planes intersecting our flowing Lagrangian tori, so we make the following definition:

Definition 4.3.1. Denote by l_b the line $\{se^{ib} : s \in \mathbb{R}\} \subset \mathbb{C}$. For $a \in (0, \pi)$, let C^b_a be a cone of opening angle a about l_b, i.e. the union of $l_{b-a/2}$ and $l_{b+a/2}$. We say that a point-symmetric pair of closed curves γ is contained in C^b_a if $\arg(\gamma(s)) \in (b - a/2, b + a/2) \cup (-b - a/2, -b + a/2)$ for all s.

Finally, we define the symmetry condition we will be using.

Definition 4.3.2. A Lagrangian L_γ is called equivariant if γ is point-symmetric and $\mathbb{Z}_2$-symmetric with respect to the real axis.

The point-symmetry is an S^1-symmetry on the level of L_γ, so the equivariance considered is an $(S^1 \times \mathbb{Z}_2)$-symmetry. The main reason for this symmetry condition is to greatly restrict the variety of singularities that can occur. Specifically, we want to have only Lawlor neck singularities occurring at the origin.

4.4 The equivariant mean curvature flow

Before proceeding to the proofs of the main theorems, we calculate the evolution equation satisfied by the profile curve γ under mean curvature flow. Despite being the governing equation for the rest of the results in the paper, we do not need the precise formulation frequently: it is only necessary for the explicit construction of various barriers. However, the derivation of the evolution equation for γ is interesting in its own right since we calculate the mean curvature of L_γ by a novel method.

Recall the fibration $\{L_\alpha\}$ by Clifford-type tori given by the fibres of the moment map

$$\mu([x:y:z]) = \frac{1}{|x|^2+|y|^2+|z|^2}\left(|x|^2,|y|^2\right).$$

The equivariant fibres are

$$L_r = \{L_{re^{i\phi}} : r > 0\}$$

and for the rest of this thesis, we denote by Ω the holomorphic volume form relative to $\{L_\alpha\}$. We first calculate the mean curvature of L_r, then we calculate the mean curvature of any other equivariant torus L_γ by calculating the relative Lagrangian angle between L_γ and L_r using Theorem 1.2.3. Recall that Theorem 1.2.3 implies that the relative Lagrangian angle $\theta = \theta_{\text{rel}}$ defined by Ω satisfies

$$H_{L_\gamma}(X) = d\theta_{\text{rel}}(X) + H_{L_r}(\pi X)$$

where π is the projection onto the tangent bundle of L_r.

We calculate the mean curvature 1-form of Clifford tori L_r indirectly. The curve

$\gamma(s) = re^{is}$ bounds a J-holomorphic disc which lifts to $\mathbb{CP}^2$ giving a disc

$$u : z \mapsto [rz : rz : 1]$$

with boundary on L_r of Maslov index 4. Cieliebak–Goldstein gives

$$-\int_{\partial D} H_{L_r} = 6\int_D \omega - 4\pi$$

since $\kappa = 6$ for $\mathbb{CP}^2$ with the Fubini–Study metric. We calculate $\int_D \omega$ directly. We have that in radial coordinates $x = r_1 e^{i\theta_1}$, $y = r_2 e^{i\theta_2}$, the Kähler form is

$$\omega = \frac{1}{\left(1+r_1^2+r_2^2\right)^2}\Big(r_1(1+r_2^2)dr_1\wedge d\theta_1 - r_1 r_2^2 dr_1\wedge d\theta_2 \\ - r_1^2 r_2 dr_2\wedge d\theta_1 + r_2(1+r_1^2)dr_2\wedge d\theta_2\Big),$$

so

$$\int_D \omega = 2\pi\int_0^r \frac{2\tilde{r}}{(1+2\tilde{r}^2)^2}\,d\tilde{r} = \pi\frac{2r^2}{1+2r^2}. \tag{4.4.1}$$

Hence using Cieliebak–Goldstein, we have

$$-\int_{\partial D} H_{L_r} = 6\pi\frac{2r^2}{1+2r^2} - 4\pi = 4\pi\left(\frac{r^2-1}{1+2r^2}\right).$$

Note that $r = 1$ is the monotone flat Clifford torus. Then by the symmetry of the tori L_r, we have that

$$H_{L_r} = -2\left(\frac{r^2-1}{1+2r^2}\right)ds \tag{4.4.2}$$

as a 1-form on L_r.

Next we calculate the relative Lagrangian angle. If $\gamma(s) = r(s)e^{i\phi(s)}$, then L_γ is given by the embedding

$$F_\gamma : (s,\alpha) \to \left[r(s)e^{i\phi(s)}e^{i\alpha} : r(s)e^{i\phi(s)}e^{-i\alpha} : 1\right],$$

so the tangent space to L_γ is spanned by

$$\frac{\partial F_\gamma}{\partial s} = \left(\left(r' + ir\phi' \right) e^{i\phi} e^{i\alpha}, \left(r' + ir\phi' \right) e^{i\phi} e^{-i\alpha} \right)$$
$$\frac{\partial F_\gamma}{\partial \alpha} = \left(ire^{i\phi} e^{i\alpha}, -ire^{i\phi} e^{-i\alpha} \right)$$

where we have identified the tangent space of $\mathbb{CP}^2$ in the coordinate patch where $z = 1$ with $\mathbb{C}^2$ in the obvious way. Note that

$$\begin{aligned}
\partial_{r_1} &= (e^{i\phi} e^{i\alpha}, 0) \\
\partial_{r_2} &= (0, e^{i\phi} e^{-i\alpha}) \\
\partial_{\theta_1} &= (ire^{i\phi} e^{i\alpha}, 0) \\
\partial_{\theta_2} &= (0, ire^{i\phi} e^{-i\alpha})
\end{aligned}$$

so

$$\frac{\partial F_\gamma}{\partial s} = r'(\partial_{r_1} + \partial_{r_2}) + \phi'(\partial_{\theta_1} + \partial_{\theta_2})$$
$$\frac{\partial F_\gamma}{\partial \alpha} = \partial_{\theta_1} - \partial_{\theta_2}$$

and hence

$$\omega \left(\frac{\partial F_\gamma}{\partial s}, \frac{\partial F_\gamma}{\partial \alpha} \right) = \omega \left(\partial_{r_1} + \partial_{r_2}, \partial_{\theta_1} - \partial_{\theta_2} \right) = 0,$$

which verifies that L_γ is Lagrangian. Furthermore, since $\partial_{r_1} + \partial_{r_2}$ and $\partial_{\theta_1} - \partial_{\theta_2}$ are tangent to L_r, we have

$$\Omega_{L_r} \left(\frac{\partial F_\gamma}{\partial s}, \frac{\partial F_\gamma}{\partial \alpha} \right) = \Omega_{L_r} \left(-r^{-1} r' J \partial_\phi + \phi' \partial_\phi, \frac{\partial F_\gamma}{\partial \alpha} \right),$$

where $\partial_\phi = \partial_{\theta_1} + \partial_{\theta_2}$ and we have used $J\partial_{\theta_i} = -r_i \partial_{r_i}$. Since $\frac{\partial F_\gamma}{\partial \alpha}$ is tangent to L_r, the Lagrangian angle θ relative to Ω_{L_r} is given by

$$\theta = \arg\left(\phi' - ir' r^{-1} \right) = -\tan^{-1}\left(\frac{r'}{r\phi'} \right)$$

and hence

$$d\theta = \frac{-r''r\phi' + r'^2\phi' + r'r\phi''}{r'^2 + r^2\phi'^2} ds. \tag{4.4.3}$$

But the Euclidean planar curvature k of γ is

$$\begin{aligned} k &= \frac{-r''r\phi' + 2r'^2\phi' + r'r\phi'' + r^2\phi'^3}{(r'^2 + r^2\phi'^2)^{3/2}} \\ &= \left(\frac{-r''r\phi' + r'^2\phi' + r'r\phi''}{(r'^2 + r^2\phi'^2)} + \phi' \right) \frac{1}{\sqrt{r'^2 + r^2\phi'^2}} \end{aligned} \tag{4.4.4}$$

We have that the projection of $\frac{\partial F_\gamma}{\partial s}$ onto L_r is

$$\pi\left(\frac{\partial F_\gamma}{\partial s}\right) = \frac{\omega\left(\frac{\partial F_\gamma}{\partial s}, J\frac{\partial F_r}{\partial s}\right)}{\omega\left(\frac{\partial F_r}{\partial s}, J\frac{\partial F_r}{\partial s}\right)} \frac{\partial F_r}{\partial s} = \phi' \frac{\partial F_r}{\partial s}$$

so we are led to conclude that

$$H_{L_r}\left(\pi\left(\frac{\partial F_\gamma}{\partial s}\right)\right) = -2\left(\frac{r^2 - 1}{1 + 2r^2}\right)\phi'. \tag{4.4.5}$$

Combining the above equations, we obtain

$$\begin{aligned} H_{L_\gamma} = d\theta + H_{L_r}(\pi(\cdot)) &= \left(k\sqrt{r'^2 + r^2\phi'^2} - \phi' - 2\left(\frac{r^2 - 1}{1 + 2r^2}\right)\phi' \right) ds \\ &= \left(k\sqrt{r'^2 + r^2\phi'^2} + \left(\frac{1 - 4r^2}{1 + 2r^2}\right)\phi' \right) ds. \end{aligned}$$

Hence we have that

$$\omega\left(\frac{\partial F_\gamma}{\partial s}, \vec{H}_{L_\gamma}\right) = k\sqrt{r'^2 + r^2\phi'^2} + \left(\frac{1 - 4r^2}{1 + 2r^2}\right)\phi',$$

but

$$\begin{aligned} \omega\left(\frac{\partial F_\gamma}{\partial s}, J\frac{\partial F_\gamma}{\partial s}\right) &= \omega\left(r'(\partial_{r_1} + \partial_{r_2}) + \phi'(\partial_{\theta_1} + \partial_{\theta_2}), r'r^{-1}(\partial_{\theta_1} + \partial_{\theta_2}) - r\phi'(\partial_{r_1} + \partial_{r_2})\right) \\ &= \left(r'^2 r^{-1} + r\phi'^2\right)\omega\left(\partial_{r_1} + \partial_{r_2}, \partial_{\theta_1} + \partial_{\theta_2}\right) \end{aligned}$$

$$= 2\frac{r'^2 + r^2\phi'^2}{(1+2r^2)^2}.$$

So we conclude that

$$\vec{H}_{L_\gamma} = \frac{1}{2}\left(1+2r^2\right)^2 \left(k + \left(\frac{1-4r^2}{1+2r^2}\right)\frac{\phi'}{\sqrt{r'^2 + r^2\phi'^2}}\right) DF_\gamma(\nu)$$

where ν is the Euclidean normal to γ in $\mathbb{C}$. Since $\langle \gamma, \nu \rangle = -r^2\phi'/|\gamma'|$, we have that the mean curvature flow of F_γ in $\mathbb{CP}^2$ induces an equivariant flow on γ given by

$$\frac{\partial \gamma}{\partial t} = \frac{1}{2}\left(1+2r^2\right)^2 \left(k - \left(\frac{1-4r^2}{1+2r^2}\right)\frac{\langle \gamma, \nu \rangle}{r^2}\right)\nu \tag{4.4.6}$$

4.5 Triangle calculations using Cieliebak–Goldstein

In order to prove the main results of this thesis, we apply the generalised Cieliebak–Goldstein theorem to certain J-holomorphic polygons with boundary on flowing Lagrangians. The most important are triangles with one vertex at the origin. Since these triangle calculations are ubiquitous and essential in the sequel, we review the methods involved here.

4.5.1 Maslov number for polygons

First, we interpret the Maslov number $\tilde{\mu}$ appearing in the Theorem 2.2.4. For Lagrangians L_i bounding a disc D We have that

$$\tilde{\mu}(D) = \frac{-1}{2\pi}\sum_i \int_{\partial D_i} d\alpha_i,$$

where α_i satisfy

$$\tau_{L_i}^2 = e^{i\alpha_i}\tau_u^2$$

as in the proof of Theorem 2.2.4. Note that if Ω is a holomorphic volume form defining Lagrangian angles θ_i for L_i by

$$\Omega_{L_i} = e^{i\theta_i}\,\mathrm{vol}_{L_i}$$

then $\alpha_i = 2\theta_i$.

In the following two examples, we calculate the Maslov number $\tilde{\mu}$ for various J-holomorphic curves in $\mathbb{CP}^2$ described by polygons in $\mathbb{C}$.

Example 4.5.1. Let γ_1, γ_2 be curves in $\mathbb{C}$ lifting to equivariant Lagrangians $L_{\gamma_1}, L_{\gamma_2}$ intersecting at two points $p_1, p_2 \subset \mathbb{C}^*$ bounding a disc $u : (D, (\partial D_1, \partial D_2)) \to (\mathbb{C}, (\gamma_1, \gamma_2))$. For ease of notation, we denote $P = u(D)$, and consider this both as a polygon in the plane $\mathbb{C}$ and also a J-holomorphic curve in $\mathbb{CP}^2$.

Let us calculate $\tilde{\mu}(P)$. There are two situations to consider. On the one hand, when P contains the origin, then the topological component of the Maslov number $\tilde{\mu}$ is 4 since that is the Maslov class of a disc with no corners. The contribution from the turning angles at the corner p_1 is given by $(-\theta_1(p_1) + \theta_2(p_1))/\pi$, which is equivalent to the difference in Euclidean Lagrangian angle between the Lagrangian planes $T_{p_1}L_1$ and $T_{p_1}L_2$. The Euclidean Lagrangian angle difference at any point p away from the origin is simply the angle between the curves in the plane, so we find that

$$\tilde{\mu}(P) = 4 - \frac{1}{\pi}(\psi_1 + \psi_2)$$

where ψ_i are the turning angles between L_{γ_1} and L_{γ_2} in the plane at p_i. On the other hand, when P does not contain the origin, we obtain

$$\tilde{\mu}(P) = 2 - \frac{1}{\pi}(\psi_1 + \psi_2)$$

in the same way.

For our second example, we now consider the case where one of the corners of P is either the origin or infinity in $\mathbb{CP}^1$. For simplicity, we assume, as will be typical of future calculations, that two of our flowing Lagrangians are parts of the minimal cones C^0_ψ.

Example 4.5.2. Let L_γ be an equivariant Lagrangian in $\mathbb{CP}^2$ intersecting the cone C^0_ψ at points p^+, p^-, see Figure 4.4, with Euclidean turning angle ξ at p^+, p^-. Consider the J-holomorphic triangle P with boundary on L_γ given by the horizontal lift of the Euclidean triangle (also denoted P) with boundary on γ, C^0_ψ and vertices at $0, p^+$ and

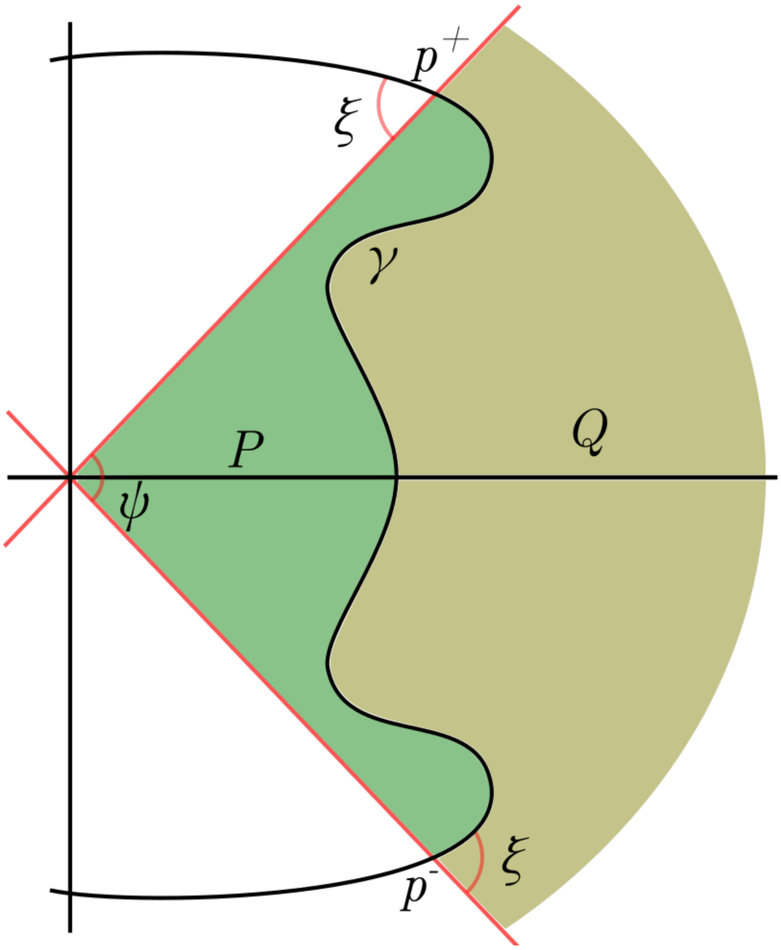

Figure 4.4: The triangles P and Q considered in Example 4.5.2

p^-.

We first calculate $\tilde{\mu}(P)$. The calculation proceeds as in Example 4.5.1 with the topological component of $\tilde{\mu}$ equal to 2. The angle contribution at the corners p^+, p^- is as above, so we have

$$\tilde{\mu}(P) = 2 - \frac{2}{\pi}\xi - A(\psi),$$

where $A(\psi)$ is some function of the opening angle at the origin to be determined.

We could calculate this directly by calculating the difference in Lagrangian angle between $l_{\psi/2}$ and $l_{-\psi/2}$. For the purposes of intuition however, we calculate indirectly using the example where $\gamma(s) = e^{is}$ is the minimal Clifford torus. We have that $\xi = \pi/2$, so

$$\tilde{\mu}(P) = 1 - A(\psi).$$

Furthermore, the area of P is given by

$$\int_P \omega = \frac{\psi}{2\pi}\frac{4\pi}{6} = \frac{\psi}{3}$$

since the area is $4\pi/6$ when $\psi = 2\pi$. Since $H_{L_\gamma} = 0$, (2.2.5) implies that

$$A(\psi) = 1 - \frac{\kappa}{\pi}\int_P \omega = \frac{1}{\pi}(\pi - 2\psi).$$

Since the contribution of ψ at the origin is independent of the choice of γ, we have that

$$\tilde{\mu}(P) = 2 - \frac{2}{\pi}\xi - \frac{1}{\pi}(\pi - 2\psi). \tag{4.5.1}$$

In the important special case where $\xi = \pi$, i.e. γ is tangent to the cone C^0_ψ at the points p^+ and p^-, the sign of $\tilde{\mu}(P)$ is controlled by the opening angle ψ. We have that

$$\tilde{\mu}(P) = -\frac{1}{\pi}(\pi - 2\psi)$$

and hence $\tilde{\mu}(P)$ is negative for $\psi < \pi/2$ and positive for $\psi > \pi/2$.

In a similar way to the above, we can calculate $\tilde{\mu}(Q)$, where Q is the complement of P in the cone C^0_ψ, see Figure 4.4. Here the contribution of the angle ψ to $\tilde{\mu}(Q)$ is

$$B(\psi) = \frac{1}{\pi}(\pi - \psi).$$

To see this, note that the area contained in the cone C^0_ψ is

$$\int_P \omega + \int_Q \omega = \frac{\psi}{2}$$

so (2.2.5) implies that

$$\tilde{\mu}(Q) = 2 - \frac{2}{\pi}(\pi - \xi) - \frac{1}{\pi}(\pi - \psi) = \frac{2}{\pi}\xi - \frac{1}{\pi}(\pi - \psi).$$

4.5.2 Evolution equations for polygons

Since they are important in the sequel, we recall the key formulae concerning H and θ. We have that

$$H = d\theta + \alpha,$$

where α is the 1-form $H_{L_r}(\pi(\cdot))$, where π is projection to the tangent bundle of L_r. Furthermore, θ defined in this way satisfies the evolution equation

$$\frac{\partial}{\partial t}\theta = \Delta\theta + d^\dagger\alpha,$$

by the same calculation that yielded (2.2.1), and the mean curvature 1-form H satisfies

$$\frac{\partial}{\partial t}H = dd^\dagger H + \kappa H.$$

Recall that for a polygon P with no corners, the Maslov number is the Maslov class and is invariant under mean curvature flow, and so we have

$$\frac{\partial}{\partial t}\int_P \omega = -\frac{1}{\kappa}\frac{\partial}{\partial t}\int_{\partial P} H = -\frac{1}{\kappa}\int_{\partial P} dd^\dagger H + \kappa H = \kappa\int_P \omega - \pi\mu(P).$$

It initially seems reasonable to conjecture then that for a polygon P with corners,

$$\frac{\partial}{\partial t}\int_P \omega = \kappa\int_P \omega - \pi\tilde{\mu}(P).$$

However, this does not hold for two reasons. Firstly, we obtain boundary terms from integrating $dd^\dagger H$. Secondly, when differentiating, we must account for potential tangential motion of the vertices of the polygon under mean curvature flow.

For these reasons, we only consider the evolution equations in the context of Example 4.5.2. We note that in this case we have that the sides of the triangle on the cone are constant angle and minimal.

To that end, let L_γ be a flowing equivariant Lagrangian, intersecting the cone C^0_ψ at points $p^\pm$, forming a triangle P as in Example 4.5.2. Initially, we assume the intersections are transverse. Writing θ for the relative Lagrangian angle of L_γ and $H = H_{L_\gamma}$ for the mean curvature 1-form, by differentiating (2.2.5) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_P \omega &= \frac{\partial}{\partial t}\left(\frac{\pi}{\kappa}\tilde{\mu}(P) - \frac{1}{\kappa}\int_\gamma H\right)\\ &= \frac{1}{\kappa}\frac{\partial}{\partial t}\left(\theta(p^-) - \theta(p^+)\right) - \frac{1}{\kappa}\frac{\partial}{\partial t}\int_\gamma H\end{aligned}$$

From each term, we obtain a normal and tangential term to account for the tangential movement of the intersection points $p^\pm$ along C^0_ψ under the flow. Writing the mean curvature flow as

$$\frac{\partial}{\partial t}X = \vec{H} + V$$

for a tangential diffeomorphism V to be determined, we have that

$$\frac{\partial}{\partial t}\left(\theta(p^\pm)\right) = \Delta\theta(p^\pm) + d^\dagger\alpha(p^\pm) + \langle\nabla\theta, V\rangle(p^\pm),$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\int_\gamma H &= \int_\gamma\left(dd^\dagger H + \kappa H\right) + \left\langle\nabla\int_\gamma H, V\right\rangle\\ &= \kappa\int_\gamma H - \Delta\theta(p^-) + \Delta\theta(p^+) - d^\dagger\alpha(p^-) + d^\dagger\alpha(p_2)\\ &\quad - \langle\nabla\theta, V\rangle(p^-) + \langle\nabla\theta, V\rangle(p^+) - \alpha(V)(p^-) + \alpha(V)(p^+)\end{aligned}$$

where we have used that $H = d\theta + \alpha$ and hence $d^\dagger H = \Delta\theta + d^\dagger\alpha$, where α is the closed 1-form on L defined by $\alpha(X) = H_{L_r}(\pi X)$. Combining the above equations and applying (2.2.5), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_P \omega &= \kappa\int_P \omega - \pi\tilde{\mu}(P)\\ &\quad + \frac{1}{\kappa}\left(-\alpha(V)(p^-) + \alpha(V)(p^+)\right).\end{aligned} \tag{4.5.2}$$

Since the intersection is transversal, we can write the tangential vector field V as $\vec{H} + V = W$, for some vector field W on L_γ tangent to C^0_ψ. The vector field W then gives the motion of $p^{\pm}$ along the cone, and we have that

$$\alpha(V) = \alpha(W - \vec{H}) = \alpha(-\vec{H}).$$

Note that while V is not well-defined when the intersection is not transversal, $\alpha(-\vec{H})$ is well-defined everywhere on L_γ. Thus it is tempting to claim that

$$\begin{aligned}\frac{\partial}{\partial t}\int_P \omega =& \kappa\int_P \omega - \pi\tilde{\mu}(P) \\ &+ \frac{1}{\kappa}\left(-\alpha(-\vec{H})(p^-) + \alpha(-\vec{H})(p^+)\right).\end{aligned} \tag{4.5.3}$$

even when the intersection is non-transversal. The most important case of this is characterised in the following lemma, where ψ is a local maximum opening angle, allowed to vary in time.

Lemma 4.5.3. *Let L_γ be an equivariant Lagrangian mean curvature flow in $\mathbb{CP}^2$ on a time interval $[t_1,t_2]$, with γ not passing through the origin. Suppose that for $t \in [t_1,t_2]$, L_γ has a local maximum opening angle $\psi(t)$ on $[t_1,t_2]$, where $\psi(t)$ is a smooth function of t. Then the triangle P defined by the cone C^0_ψ and γ, with vertices at $p^\pm$ and the origin, satisfies*

$$\frac{d}{dt}\int_P \omega \le \kappa\int_P \omega + (\pi - 2\psi) \tag{4.5.4}$$

Proof. Let $\gamma(s)$ be parametrised by some variable s. Then there exists a smooth function $S(t)$ such that $\gamma(S(t))$ attains the maximum opening angle $\psi(t)$.

Let $A(s,t) = \int_{P_s}\omega$, where P_s is the triangle intersecting γ at $\gamma(s)$. Here, the integral is the signed integral of ω, see Figure 4.5. Then we have to calculate the time-derivative of A at $S(t)$ for $t \in (t_1,t_2)$. By choosing a sufficiently small time neighbourhood $(t_-,t_+) \subset (t_1,t_2)$ of t, we can find a time-independent space neighbourhood (s_-,s_+) of $S(t)$ for all t such that $\gamma(s)$ intersects the cone transversally for all $s \neq S(t)$.

For any fixed opening angle χ with transversal intersections with γ at $p^\pm_\chi$, we have that

$$\frac{\partial}{\partial t}\int_{P_\chi}\omega = \kappa\int_{P_\chi}\omega - \pi\tilde{\mu}(P_\chi) + \frac{1}{\kappa}\left(-\alpha(-\vec{H})(p^+_\chi) + \alpha(-\vec{H})(p^-_\chi)\right),$$

where P_χ is the triangle of opening angle χ, again calculated with sign. Now allowing

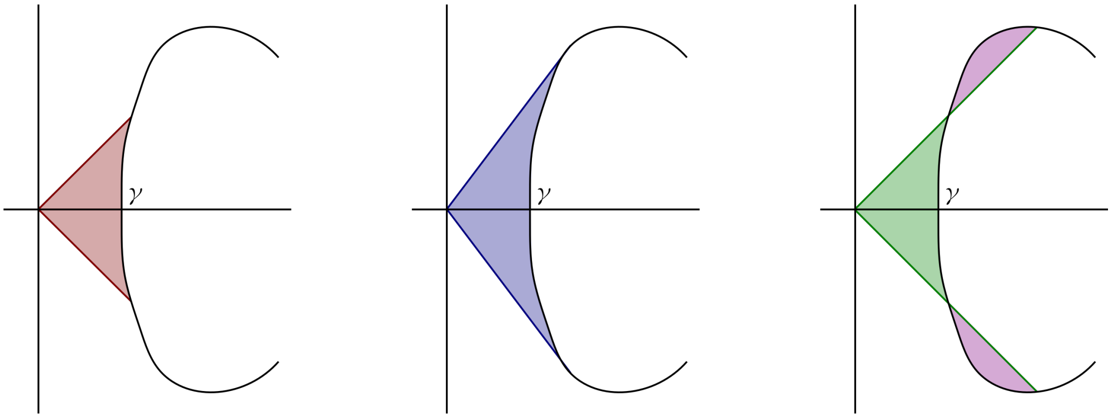

Figure 4.5: Three triangles P_s, ordered with increasing s. In the right diagram, the area is counted with sign, i.e. the green area is counted positively and the purple area negatively.

that the opening angle $\chi = \chi(s,t)$ may evolve with s,

$$\frac{d}{dt}A(s,t) = \frac{\partial}{\partial t}\int_{P_\chi} \omega + \frac{d\chi}{dt}(s,t)\frac{\partial A}{\partial \chi}(s,t)$$

and taking limits as $s \to S(t)$ gives

$$\begin{aligned}\frac{d}{dt}A(S(t),t) =& \kappa\int_{P_\chi} \omega - \pi\tilde{\mu}(P_\chi) + \frac{1}{\kappa}\left(-\alpha(-\vec{H})(p_\chi^+) + \alpha(-\vec{H})(p_\chi^-)\right)\\ &+ \frac{d\chi}{dt}(S(t),t)\frac{\partial A}{\partial \chi}(S(t),t) + \frac{dS}{dt}(t)\frac{\partial A}{\partial s}(S(t),t).\end{aligned}$$

But since $S(t)$ is a local maximum of the area by assumption, we have that

$$\frac{\partial A}{\partial s}(S(t),t) = 0.$$

Furthermore, the maximum opening angle is decreasing in time, so

$$\frac{d\chi}{dt}(S(t),t) \leq 0,$$

and A is always increasing in χ for $\chi < \psi$, so

$$\frac{d\chi}{dt}(S(t),t) \geq 0.$$

Finally,

$$-\alpha(-\vec{H})(p^+) + \alpha(-\vec{H})(p^-) < 0$$

since the direction of the mean curvature is fixed by the assumption that $p^{\pm}$ are at the maximum opening angle. Since the Maslov number satisfies $\pi\tilde{\mu}(P) = -(\pi - 2\psi)$, we conclude that

$$\frac{d}{dt}A(S(t),t) \leq \kappa \int_{P_\chi} \omega + (\pi - 2\psi),$$

as desired. □

4.6 Minimal equivariant Lagrangians

From equation (4.4.6), any minimal equivariant Lagrangian must satisfy

$$k - \left(\frac{1-4r^2}{1+2r^2}\right)\frac{\langle \gamma, \nu \rangle}{r^2} = 0. \tag{4.6.1}$$

Away from the origin, equation (4.6.1) is a non-linear 2nd order ODE. Given any point $x \in \mathbb{C}$ and an initial velocity $v \in T_x\mathbb{C}$, there is a unique local solution to (4.6.1) passing through x with velocity v. The proof is identical to the equivalent statement for existence and uniqueness of geodesics.

Two classes of solutions to (4.6.1) are immediately apparent. First, either from the derivation of (4.6.1) or by direct calculation, one can see that the Clifford torus $L_1 := L_{e^{is}}$ given by the unit circle is a minimal submanifold. Second, any straight line through the origin $l_b = \{se^{ib} : s \in \mathbb{R}\}$ has $k = 0$ and $\langle l_b, \nu \rangle = 0$, and hence gives a minimal submanifold of $\mathbb{CP}^2$, topologically a real projective plane.

Combining the second class of solutions with the prior observation that solutions to equation (4.6.1) are unique, we realise the following: If γ is a solution to (4.6.1) with $\langle \gamma, \nu \rangle = 0$ at a point $p \in \mathbb{C}^*$ (or equivalently, a point $p \in \mathbb{C}^*$ with relative Lagrangian angle $\pm\pi/2$), then γ is a line l_b.

Let us consider minimal Lagrangians that can be written as graphs over (sections of) the unit circle; that is to say Lagrangians given by point-symmetric curves $\gamma(s) = r(s)e^{is}$

with $r(s) \in (0,\infty)$. From (4.6.1), we have that r satisfies

$$-r''r + 2r'^2 + r^2 + \left(\frac{1-4r^2}{1+2r^2}\right)(r'^2 + r^2) = 0.$$

Rearranging, we obtain

$$-r''r + \left(\frac{3}{1+2r^2}\right)r'^2 + 2\left(\frac{1-r^2}{1+2r^2}\right)r^2 = 0. \tag{4.6.2}$$

We make the substitution

$$f(s) = \log(r(s)).$$

Then we have

$$-f'' + 2\frac{1-r^2}{1+2r^2}f'^2 + 2\frac{1-r^2}{1+2r^2} = 0. \tag{4.6.3}$$

We aim to find a first integral of the equation. To that end, let $y = f'$. Then

$$f'' = y\frac{dy}{df},$$

hence

$$\frac{dy}{df} - 2\frac{1-r^2}{1+2r^2}y = 2\frac{1-r^2}{1+2r^2}y^{-1}.$$

Making a further substitution $u = y^2$, we have

$$\frac{du}{df} - 4\frac{1-r^2}{1+2r^2}u = 4\frac{1-r^2}{1+2r^2}, \tag{4.6.4}$$

Denoting

$$P(f) = 4\frac{1-r^2}{1+2r^2} = 4\frac{1-e^{2f}}{1+2e^{2f}},$$

we have that (4.6.4) is solved by use of an integrating factor, giving

$$f'^2 = u = e^{\int P(f)df}\int P(f)e^{-\int P(f)df}df + Ce^{\int P(f)df},$$

where C is a constant determined by the initial conditions. Calculating explicitly, we have

$$f'^2 = C\frac{e^{4f}}{(1+2e^{2f})^3} - 1 =: B(f,C). \tag{4.6.5}$$

Explicit calculation verifies that this is indeed a solution of equation (4.6.3).

Given any constant $C > 0$, there is a constant $R > 0$ such that $B(f,C) < 0$ for all $|f| > R$. Hence every solution of (4.6.3) has f bounded, and therefore any solution to (4.6.2) has r bounded away from 0 and infinity.

To find the minima and maxima of a solution to (4.6.2), we have to find zeroes of $B(f,C)$. Letting $x = e^{2f}$, we see that the zeroes of $B(f,C)$ are determined by the positive real roots of the cubic

$$P_C(x) = 8x^3 + (12-C)x^2 + 6x + 1.$$

The discriminant of $P_C(x)$ is

$$\Delta = 4C^2(C-27).$$

There are then 3 cases to consider. Firstly, when $C = 27$, $P_C(1) = 0$ is a repeated root: this corresponds to the minimal Clifford torus L_1. The third root is $-1/8$, which does not correspond to a root of $B(f,C)$. Secondly, When $C < 27$, $P_C(x)$ has 1 real root and two complex roots. Furthermore, in this case the real root is negative by applying Descartes' rule of signs to $P_C(-x)$, implying that $B(f,C)$ has no roots for $C < 27$.

Finally, if $C > 27$, $P_C(x)$ has 3 distinct real roots. Applying Descartes' rule of signs to $P_C(x)$ and $P_C(-x)$ implies that $P_C(x)$ in this case has two positive real roots $r_1 < r_2$ and one negative real root. The two positive roots determine zeroes of $B(f,C)$. We do not find the roots explicitly, though this could of course be done by the formula for roots of a cubic. However, by examining $B(f,C)$ directly, we can see that $B(f,C)$ has 1 positive root and 1 negative root, which implies that

$$0 < r_1 < 1 < r_2 < \infty.$$

Note that as $C \to \infty$, $r_1 \to 0$ and $r_2 \to \infty$ monotonically.

Next, we approach the question of periodicity of solutions to (4.6.2). Since solutions can only have maxima at r_2 and minima at r_1 and are strictly bounded between r_1 and r_2, they must oscillate between the two with some period ψ_C. Note the solution must be periodic since any solution to (4.6.2) is uniquely determined by the Cauchy boundary condition $r(s_0) = r_1$ (or r_2) and $r'(s_0) = 0$.

The period ψ_C is given by solving (4.6.5). We have that

$$\psi_C = 2\int_{\log r_1}^{\log r_2} \sqrt{\frac{(1+2e^{2f})^3}{Ce^{4f} - (1+2e^{2f})^3}}\, df = \int_{r_1}^{r_2} \sqrt{\frac{(1+2r^2)^3}{Cr^4 - (1+2r^2)^3}}\, \frac{1}{r}\, dr,$$

where $\log r_1$ and $\log r_2$ are the positive real roots of $B(f,C)$. This integral cannot be easily evaluated by standard methods. However, with standard methods, we can determine lower bounds for ψ_C for specific cases. We present $C = 54$ as an example since it will be useful in the sequel.

Example 4.6.1. Let $C = 54$. Then the roots of

$$8x^3 + (12-54)x^2 + 6x + 1 = 0$$

are

$$r_0^2 = \frac{1}{2}(5-3\sqrt{3}), \quad r_1^2 = \frac{1}{4}, \quad r_2^2 = \frac{1}{2}(5+3\sqrt{3}).$$

So making the substitution $x = r^2$, we have that

$$\psi_{54} = \int_{1/4}^{\frac{1}{2}(5+3\sqrt{3})} A(x) \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)\left(\frac{1}{2}(5+3\sqrt{3})-x\right)}}\, dx,$$

where

$$A(x) = \frac{(1+2x)^{3/2}}{2x\sqrt{x-\frac{1}{2}(5-3\sqrt{3})}}.$$

Note that $A(x)$ is decreasing in x for $x > 0$. So

$$\psi_{54} > A\left(\frac{1}{2}(5+3\sqrt{3})\right)\int_{1/4}^{\frac{1}{2}(5+3\sqrt{3})} \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)\left(\frac{1}{2}(5+3\sqrt{3})-x\right)}}\, dx = A\left(\frac{1}{2}(5+3\sqrt{3})\right)\pi,$$

where we have used that

$$\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}\,dx = \pi,$$

as can be shown using the substitution $x = t + (a+b)/2$; letting $y = (a-b)/2$, we have that

$$\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}\,dx = \int_{-y}^{y} \frac{1}{\sqrt{y^2-t^2}}\,dt = \sin^{-1}(1) - \sin^{-1}(-1) = \pi.$$

We have that

$$A\left(\frac{1}{2}(5+3\sqrt{3})\right)^2 = \frac{(6+3\sqrt{3})^3}{(5+3\sqrt{3})^2 3\sqrt{3}} = \frac{702+405\sqrt{3}}{207+156\sqrt{3}} > \frac{9}{4}.$$

So $\psi_{54} > 3\pi/2$.

We aim instead to analyse the limiting behaviour as $C \to \infty$, illustrated in Figure 4.6. We show the following:

Lemma 4.6.2. *The period ψ_C converges to $3\pi/2$ as $C \to \infty$.*

Proof. Let $\gamma_C(s) = r(s)e^{is}$ be a minimal equivariant Lagrangian. Without loss of generality, we suppose that r satisfies (4.6.2) subject to the initial condition $r'(0) = 0$ with $r(0) = r_1 < 1$ the minimal value being determined by $C = (1+2r_1^2)^3/r_1^4 > 27$.

Denote by ψ_C^- the inner period, i.e. the period where $r(s) \leq 1$. Similarly, denote by ψ_C^+ the outer period. Note that $\psi_C = \psi_C^- + \psi_C^+$.

First, we show that ψ_C^- is strictly greater than $\pi/2$ for all C with $\psi_C^- \to \pi/2$ as $C \to \infty$. To do so, we need two geometric inequalities that relate the inner period ψ_C^- to the minimum value r_1. See Figure 4.7 for the following setup. Denote by P the J-holomorphic biangle lying inside B_1 with sides on γ_C and L_1, and vertices at the intersection points $p^\pm$ between γ_C and L_1. By assumption, we have that $p^\pm = e^{i\psi_C^-/2}$. Let A denote the J-holomorphic quadrangle with sides on L_1, L_{r_1} and the cone $C^0_{\psi_C^-}$. Finally, denote by B the J-holomorphic triangle with vertices at p^-, p^+ and the minimum value $\gamma_C(0) = r_1$, and sides on L_1 and $\eta^\pm$, where $\eta^\pm$ is the radial straight line connecting $\gamma_C(0)$

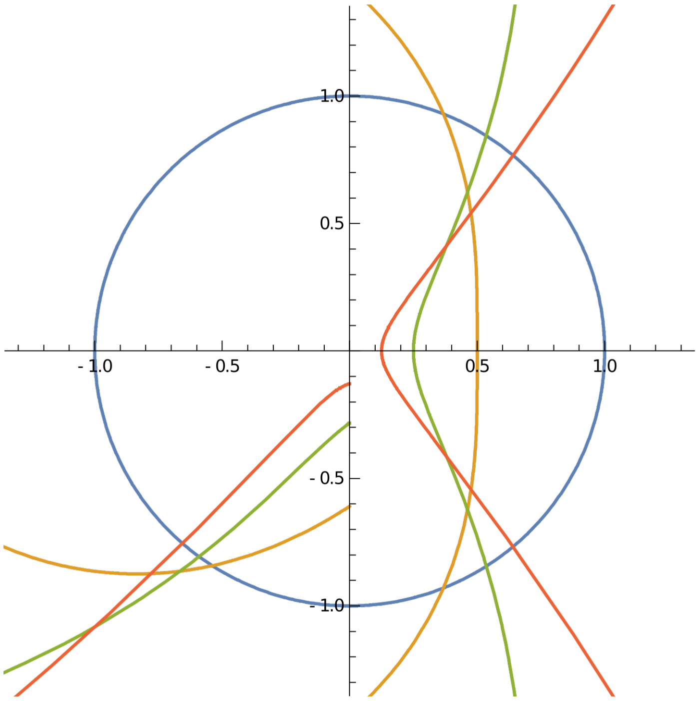

Figure 4.6: The inner period converges to $\pi/2$ and the outer period converges to π as $C \to \infty$.

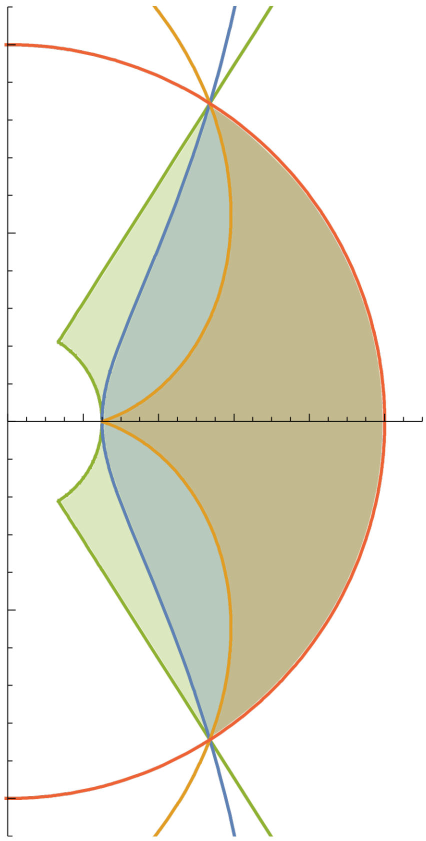

Figure 4.7: The area of P (the blue region) is bounded above by the area of A (the green region) and bounded below by the area of B (the orange region).

to $p^{\pm}$, i.e.

$$\eta^{\pm}(s) = \left(2\frac{1-r_1}{\psi_C^-}s + r_1\right)e^{\pm is}.$$

Then we have the geometric inequalities

$$\kappa\int_A \omega > \kappa\int_P \omega > \kappa\int_B \omega. \tag{4.6.6}$$

The first inequality is immediate. For the second inequality, we have that (4.6.2) implies that $r''(s) > 0$ for $r \leq 1$, and hence r is convex as a function of s for $r < 1$.

Since γ_C is minimal and the relative Lagrangian angle for γ_C is given by $\tan^{-1}(r'/r)$, we have that

$$\kappa\int_P \omega = \pi\tilde{\mu}(P) = 2\pi - 2\tan^{-1}\left(\sqrt{\frac{C}{27}-1}\right) = 2\pi - 2\tan^{-1}\left(\sqrt{\frac{1}{27}\frac{(1+2r_1^2)^3}{r_1^4}-1}\right).$$

We can calculate $\kappa\int_A \omega$ using Cieliebak–Goldstein. We have that $\tilde{\mu}(A) = 0$, so (4.4.2) gives

$$\kappa\int_A \omega = -\frac{\psi_C^-}{2\pi}\int_{L_{r_1}} H_{L_{r_1}} = 2\psi_C^-\frac{1-r_1^2}{1+2r_1^2}.$$

Therefore (4.6.6) implies

$$\psi_C^- > \frac{1+2r_1^2}{1-r_1^2}\left(\pi - \tan^{-1}\left(\sqrt{\frac{1}{27}\frac{(1+2r_1^2)^3}{r_1^4}-1}\right)\right),$$

so $\psi_C^- > \pi/2$ for all $C > 27$ and

$$\lim_{C\to\infty}\psi_C^- = \lim_{r_1\to 0}\psi_C^- \geq \pi/2. \tag{4.6.7}$$

Instead of using Cieliebak–Goldstein to calculate $\kappa\int_B \omega$, it is simpler to calculate directly. Letting $a = 2(1-r_1)(\psi_C^-)^{-1}$ for ease of notation, we have that

$$\begin{aligned}\kappa\int_B \omega &= 2\kappa\int_0^{\psi_C^-/2}\int_{a\phi+r_1}^1 \frac{2r}{(1+2r^2)^2}\,dr d\phi \\ &= \kappa\int_0^{\psi_C^-/2}\left[-\frac{1}{1+2r^2}\right]_{a\phi+r_1}^1 d\phi\end{aligned}$$

$$
\begin{aligned}
&= \kappa \int_0^{\psi_C^-/2} -\frac{1}{3} + \frac{1}{1+2r_1^2+4ar_1\phi+2a^2\phi^2}\,d\phi \\
&= \kappa \left[-\phi/3 + \frac{1}{\sqrt{2}a} \tan^{-1}\left(\sqrt{2}\,(a\phi+r_1)\right) \right]_0^{\psi_C^-/2} \\
&= \psi_C^- \left(-1 + \frac{3}{\sqrt{2}(1-r_1)} \left(\tan^{-1}\left(\sqrt{2}\right) - \tan^{-1}\left(\sqrt{2}r_1\right) \right) \right) \\
&= \psi_C^- \left(-1 + \frac{3}{\sqrt{2}(1-r_1)} \tan^{-1}\left(\frac{\sqrt{2}(1-r_1)}{1+2r_1} \right) \right),
\end{aligned}
$$

where we have used the equality

$$\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

in the final line. Therefore (4.6.6) implies that

$$\psi_C^- < \frac{\left(2\pi - 2\tan^{-1}\left(\sqrt{\frac{1}{27}\frac{(1+2r_1^2)^3}{r_1^4}-1}\right)\right)}{-1+\frac{3}{\sqrt{2}(1-r_1)}\tan^{-1}\left(\frac{\sqrt{2}(1-r_1)}{1+2r_1}\right)}.$$

Since $\lim_{x\to 0}\tan^{-1}(x)/x = 1$, we have that

$$\lim_{C\to\infty}\psi_C^- = \lim_{r_1\to 0}\psi_C^- \le \pi/2. \tag{4.6.8}$$

Combining (4.6.7) and (4.6.8) gives

$$\lim_{C\to\infty}\psi_C^- = \lim_{r_1\to 0}\psi_C^- = \pi/2.$$

The proof that

$$\lim_{C\to\infty}\psi_C^+ = \lim_{r_2\to\infty}\psi_C^+ = \pi$$

with $\psi_C^+ < \pi$ for all $C > 27$ is largely similar, with one additional complication. Letting P be the biangle lying outside B_1, and A and B constructed similarly to above, we claim that if we have the geometric inequalities (4.6.6), then the result follows. Indeed, if the geometric inequalities held, the proof is identical up to small changes in the calculation. The first inequality clearly holds, and so we obtain in the same way as above that $\psi_C^- < \pi$

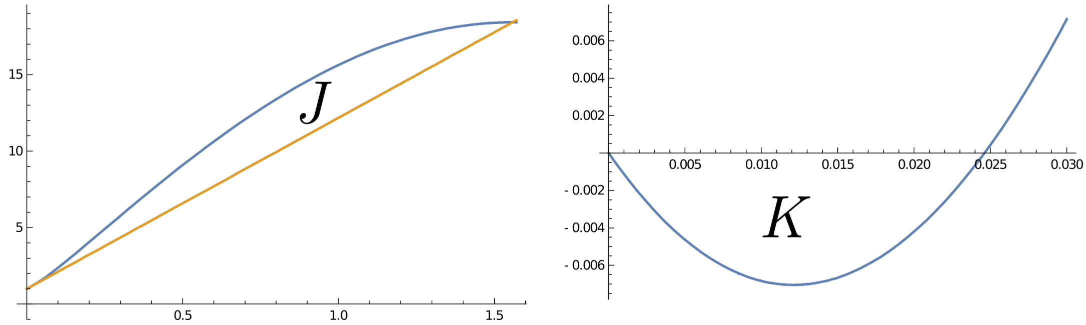

Figure 4.8: The radius $r(s)$ of a minimal curve γ_C in blue compared with the radius of a radial line η in orange. The region J is labelled. The region K is so small as to not be visible in the left diagram, but is visible on a smaller scale, see the right diagram where the difference between the minimal curve and the radial line is plotted for small s.

for all $C > 27$ and

$$\lim_{C\to\infty} \psi_C^- = \lim_{r_1\to 0} \psi_C^- \leq \pi.$$

Although the second inequality in (4.6.6) does in fact hold, it is not immediately apparent why. Above, we were able to use the fact that $r'' > 0$ for $r \leq 1$ to derive the second inequality. However, r is neither convex nor concave for $r \geq 1$, so we need a more advanced method.

It suffices then to prove that for sufficiently large $C > 27$,

$$\kappa \int_P \omega > \kappa \int_B \omega.$$

Consider then the set-up depicted in Figure 4.8, where we have plotted γ_C as a graph over s, assuming now that the intersection with the $r = 1$ line occurs at $s = 0$ for simplicity. We want to show that the area under γ_C (which is equal to $\int_P \omega/2$) is greater than the area under the area under the radial straight line $\eta(s)$, which as a radial function of s is given by $2s(r_2 - 1)/\psi_C^+ + 1$. This amounts to showing that the area $\int_J \omega$ is greater than the area $\int_K \omega$, where J and K are the depicted regions. Since γ_C is by assumption minimal, Cieliebak–Goldstein gives

$$\kappa \int_J \omega - \kappa \int_K \omega = -\int_\eta H_{L_\eta} + \zeta - \xi,$$

where ξ and ζ are the interior angles at the corners (note the Maslov classes of J and K

have cancelled significantly).

Let $a = 2(r_2 - 1)/\psi_C^+$ for ease of notation. Then we have that $\zeta = \tan^{-1}(a/r_2)$ and

$$\xi = \tan^{-1}(a) - \tan^{-1}\left(\sqrt{\frac{1}{27}\frac{(1+2r_2^2)^3}{r_2^4} - 1}\right).$$

So

$$\zeta - \xi = -\tan^{-1}\left(\frac{r_2 a - a}{r_2 + a^2}\right) - \tan^{-1}\left(\sqrt{\frac{1}{27}\frac{(1+2r_2^2)^3}{r_2^4} - 1}\right).$$

Then

$$\lim_{C\to\infty}(\zeta - \xi) = -\tan^{-1}\left(\frac{\psi_C^+}{2}\right) - \frac{\pi}{2}.$$

We have that

$$H_{L_\eta} = \frac{a^2}{a^2 - r^2} - 2\frac{r^2 - 1}{1 + 2r^2},$$

where $r = as + 1$. Then

$$\begin{aligned}
\int_\eta H_{L_\eta} &= \int_0^{\psi_C^+/2} \frac{a^2}{a^2 + 1 + 2as + a^2s^2} - \frac{2a^2s^2 + 4as}{3 + 4as + 2a^2s^2}\, ds \\
&= \left[-s + \tan^{-1}\left(\frac{1}{a} + s\right) + \frac{3\sqrt{2}}{2a}\tan^{-1}\left(\sqrt{2}(1 + as)\right)\right]_0^{\psi_C^+/2} \\
&= -\psi_C^+/2 + \tan^{-1}\left(\frac{1}{a} + \psi_C^+/2\right) - \tan^{-1}\left(\frac{1}{a}\right) \\
&\quad + \frac{3\sqrt{2}}{2a}\left(\tan^{-1}\left(\sqrt{2}(1 + a\psi_C^+/2)\right) - \tan^{-1}\left(\sqrt{2}\right)\right)
\end{aligned}$$

We claim that $a \to \infty$ as $C \to \infty$. We postpone the proof of this in favour of completing the rest of the proof. We have then that

$$\lim_{C\to\infty}\left(-\int_\eta H_{L_\eta} + \zeta - \xi\right) = \lim_{C\to\infty}\left(\psi_C^+/2 + \tan^{-1}\left(\psi_C^+/2\right) - \tan^{-1}\left(\frac{\psi_C^+}{2}\right) - \frac{\pi}{2})\right) > 0,$$

since we know that $\psi_C^+ > \pi$ for all $C > 27$. Thus the inequality

$$\kappa\int_P \omega > \kappa\int_B \omega$$

does hold for sufficiently large $C > 27$.

It remains to prove the above claim that $a \to \infty$ as $C \to \infty$, or equivalently, to prove that ψ_C^+ is bounded for all $C > 27$. To do so, it suffices to use a much weaker inequality than the above. As observed above, r is convex for $r < R$ and concave for $r > R$, where $R > 1$ is a radius determined by a root of the equation

$$r''r = \frac{3}{1+2r^2}r'^2 + 2\frac{1-r^2}{1+2r^2}r^2 = 0.$$

Since $r'^2/r^2 = f'^2 = Cr^4(1+2r^2)^{-3}$, R is a root of the equation

$$2r^2(1+2r^2)^3 + (1+2r^2)^3 - 3Cr^4 = 0.$$

For $C > 27$, the above equation has exactly one root greater than 1, namely

$$R = \sqrt{\frac{1}{8}\left(-4+\sqrt{3C}+\sqrt{C\sqrt{3}-8\sqrt{C}}\right)}.$$

Note that $R \to \infty$ as $C \to \infty$. To derive a contradiction, choose $C > 27$ sufficiently large that $\psi_C^+ > 4\pi$ and $R > 4$. Since r is convex on the interval $[1,R)$ and $\sqrt{C/27-1} \to \infty$ as $C \to \infty$, we can also choose C sufficiently large that the period of time that $r < 4$ is less than 2π. Then $\int_P \omega$ is bounded below by the area of the annulus $A(1,4)$, which is strictly bigger than $\pi/6$. So

$$\kappa \int_P \omega > \pi,$$

but Cieliebak–Goldstien gives that

$$\kappa \int_P \omega < \pi,$$

a contradiction. This completes the proof. □

We can now prove the main theorem of this section.

Theorem 4.6.3. *There exists a countably infinite family of complete immersed minimal equivariant Lagrangians. In particular, for any $R > 0$, there exists a complete immersed minimal equivariant Lagrangian L_γ with $\min_{L_\gamma} r \leq R$.*

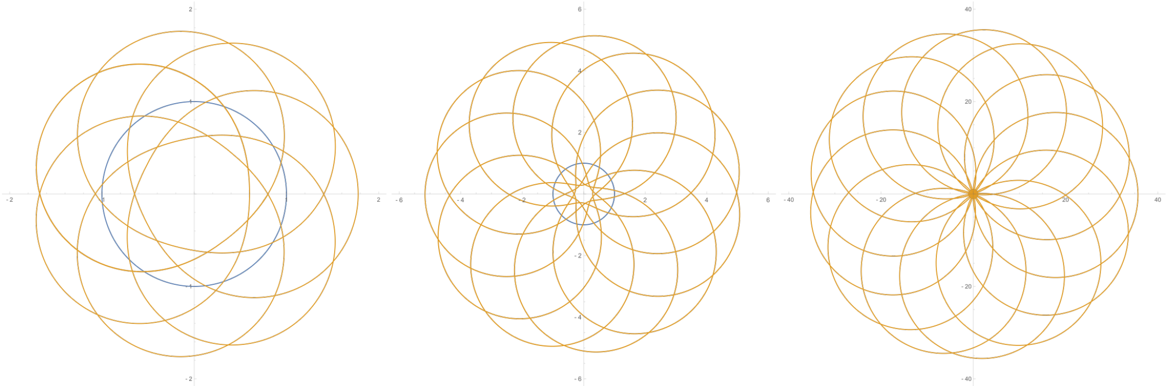

Figure 4.9: Complete immersed minimal equivariant Lagrangians of varying period. The corresponding integers pairs are $(m,k) = (7,6)$, $(11,9)$ and $(14,10)$.

Proof. Note that the period ψ_C of a solution γ_C to (4.6.2) depends continuously upon the initial condition. Since $\psi_C \to 3\pi/2$ by Lemma 4.6.2 and by Example 4.6.1 $\psi_C > 3\pi/2$ for some $C > 27$, we have that there exists $\delta > 0$ such that for every $\psi \in (3\pi/2, 3\pi/2 + \delta)$ there exists a $C > 27$ such that $\psi_C = \psi$. In particular, we can find infinitely many integer pairs (m,k) and values $C(m,k) > 27$ such that

$$m\psi_{C(m,k)} = 2\pi k.$$

Then the minimal equivariant Lagrangians $\gamma_{C(m,k)}$ described by $C(m,k)$ are complete immersed minimal equivariant Lagrangian. This is an infinitely large family of unique solutions since for every sufficiently large prime k, we can obtain at least one solution.

Since this argument also applies to any $\delta' < \delta$, we can construct L_γ satisfying the second part of the theorem. $\square$

Figure 4.9 illustrates the spirograph-like shape of the complete immersed minimal equivariant Lagrangians.

While we are on the subject, we prove one final property of the minimal surfaces which will be useful in proving Lemma 4.7.5. The idea of the proof is similar to Lemma 4.6.2, but a slightly different geometric estimate is required. Instead of estimating using a disc B with boundary on a radial straight line, we use a disc B with boundary on a Euclidean straight line, see Figure 4.10

Proposition 4.6.4. *Let $\gamma_C(s) = r(s)e^{is}$, $C > 27$, be a solution to equation (4.6.2) with*

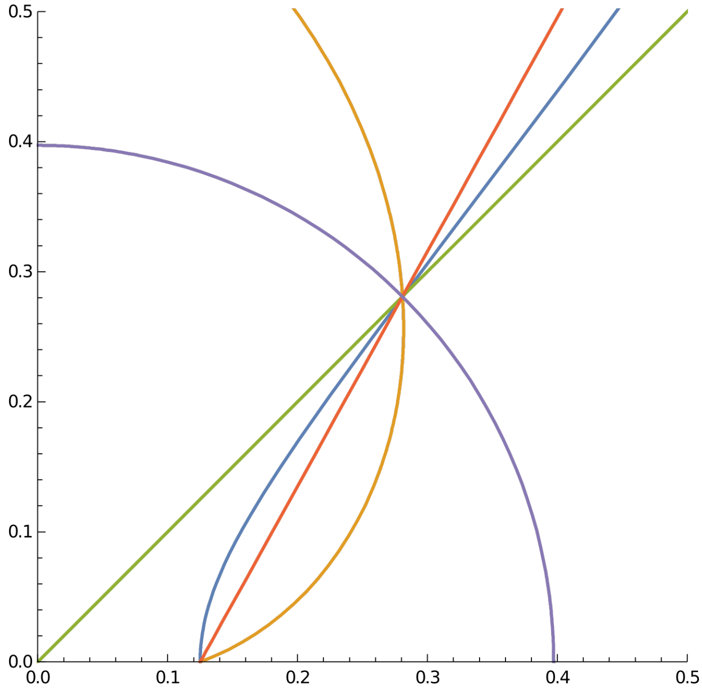

Figure 4.10: Radial straight lines (orange) used in Lemma 4.6.2 compared to the Euclidean straight line (red) used in Proposition 4.6.4. Both give lower bounds for the region contained between γ (blue) and the circle of radius R_C (purple).

initial condition $r'(0) = 0$ with $r(0) = r_1 < 1$. Let $R_C := r(\pi/4)$, i.e. the radius of intersection with the cone $C^0_{\pi/2}$. Then $R_C \to 0$ as $C \to \infty$.

We need a lemma to prove Proposition 4.6.4, which guarantees the Euclidean straight lines give lower bounds for sufficiently large $C > 27$.

Lemma 4.6.5. *Let γ_C be as in the statement of Proposition 4.6.4. Then $R_C < 1/2$ for C sufficiently large.*

The idea of the proof is to find a subsolution (a hyperbola) with the desired behaviour, and then use the comparison principle to obtain the result.

Proof. Consider the hyperbolas $\gamma_{a,c}$ given by

$$a^2x^2 - y^2 = c^2$$

for constants $a, c > 0$ to be determined. For $a > 1$, $\gamma_{a,c}$ is asymptotic to $C^0_{\pi/2+2\varepsilon(a)}$ for some $\varepsilon(a) > 0$, with $\varepsilon(a) \to 0$ as $a \to 1$. Parametrising by the x coordinate, the part of

$\gamma = \gamma_{a,c}$ in the positive quadrant is given by

$$\gamma(x) = (x, \sqrt{a^2x^2 - c^2}),$$

for $x \geq c/a$, which has unit normal

$$\nu = \frac{1}{\sqrt{a^2x^2 + a^4x^2 - c^2}} \left(\frac{-a^2x}{\sqrt{a^2x^2 - c^2}}, 1 \right).$$

The planar curvature k is then

$$k = \frac{y''}{(1+y'^2)^{3/2}} = -\frac{c^2a^2}{(a^2x^2 + a^4x^2 - c^2)^{3/2}},$$

and

$$\langle \gamma, \nu \rangle = \frac{-c^2}{\sqrt{a^2x^2 + a^4x^2 - c^2}}.$$

Hence we have that

$$(1+2r^2)k - (1-4r^2)\frac{\langle \gamma, \nu \rangle}{r^2} > \frac{1}{(a^2x^2 + a^4x^2 - c^2)^{3/2}} \Big((1+2r^2)c^2a^2 + (1-4r^2)c^2 \Big),$$

for $r < 1/2$. It follows that

$$(1+2r^2)k - (1-4r^2)\frac{\langle \gamma, \nu \rangle}{r^2} = \frac{c^2}{B} \left(-(r^2 + 2r^4)a^2 + (1-4r^2)(a^2r^2 + c^2(a^2-1)) \right),$$

where $B = \sqrt{a^2x^2 + a^4x^2 - c^2} = \sqrt{a^2r^2 + c^2(a^2-1)}$. Simplifying further and using $r^2 \geq c^2/a^2$, we have

$$\begin{aligned}(1+2r^2)k - (1-4r^2)\frac{\langle \gamma, \nu \rangle}{r^2} &\leq \frac{c^4}{B} \left(-\frac{6c^2}{a^2} + (a^2-1)(1-4r^2) \right) \\ &\leq -\frac{2c^2}{a^2} + a^2 - 1 - 4c^2.\end{aligned}$$

Solving the resulting quadratic, we find that we have

$$(1+2r^2)k - (1-4r^2)\frac{\langle \gamma, \nu \rangle}{r^2} \leq 0$$

for all $a > 1$ satisfying

$$1 < a^2 \leq \left(2c^2 + \frac{1}{2}\right) + \sqrt{\left(2c^2 + \frac{1}{2}\right)^2 + 2c^2}.$$

On the other hand, we have that $\gamma_{a,c}$ intersects the cone $C^0_{\pi/2}$ when $(a^2-1)x^2 = c^2$, which occurs at radius R with

$$R^2 = \frac{2c^2}{a^2-1}.$$

So for

$$a^2 = \left(2c^2 + \frac{1}{2}\right) + \sqrt{\left(2c^2 + \frac{1}{2}\right)^2 + 2c^2},$$

we have that

$$R^2 = 2c^2\left(\left(2c^2 - \frac{1}{2}\right) + \sqrt{\left(2c^2 + \frac{1}{2}\right)^2 + 2c^2}\right)^{-1},$$

which converges to 1/3 as $c \to 0$. So we can choose $a > 1$ and $c > 0$ such that $R < 1/2$.

Let C be any constant such that γ_C has $r_1 \leq c$. We claim this implies that γ_C intersects the cone $C^0_{\pi/2}$ at a radius less than $R < 1/2$. Suppose not. Then γ_C intersects $\gamma_{a,c}$ at two points inside the cone $C^0_{\pi/2}$. Consider the quasilinear elliptic operator $Q(f)$ given by

$$Q(f) = -f'' + 2\frac{1-r^2}{1+2r^2}f'^2 + 2\frac{1-r^2}{1+2r^2}$$

where $f = \log r$, see equation (4.6.3). Let f_C and $f_{a,c}$ be the logarithms of the radius functions of γ_C and $\gamma_{a,c}$ respectively. Then we have that $Q(f_C) = 0$ and $Q(f_{a,c}) < 0$. Furthermore, we have that for $s \in [-S,S] \subset (-\pi/4, \pi/4)$, $f_C(s) \leq f_{a,c}(s)$ with $f_C(S) = f_{a,c}(S)$. Since

$$\partial_r\left(\frac{1-r^2}{1+2r^2}\right) < 0$$

for all r, we can apply the comparison principle for quasilinear elliptic operators to deduce that

$$f_C(s) \geq f_{a,c}(s)$$

for all $s \in [-S,S]$, a contradiction. □

Proof of Proposition 4.6.4. By Lemma 4.6.5, we have that $r_1 < R_C < 1/2$ for sufficiently large $C > 27$, and hence $r(s) < 1/2$ for all $s \in [-\pi/4, \pi/4]$. Consider now γ as a graph over the y-axis, i.e. $\gamma(y) = x(y) + iy$ for y in an interval I containing 0. Since $x = r(s)\cos s$ and $y = r(s)\sin s$, simple application of the chain rule gives that

$$\frac{d^2x}{dy^2} = \frac{1}{(r'\sin s + r\cos s)^3}\left(r''r - 2r'^2 - r^2\right).$$

Using (4.6.2), we have that

$$\frac{d^2x}{dy^2} = \frac{1}{(r'\sin s + r\cos s)^3}\left(\frac{1-4r^2}{1+2r^2}\right)\left(r'^2 + r^2\right),$$

which is greater than 0 for $r < 1/2$. So $x(y)$ is convex as a function of y. Hence the Euclidean straight line $\eta^{\pm}$ connecting the minimum value r_1 with $R_C e^{\pm i\pi/4}$ does not intersect γ_C for $s \in (0, \pi/4)$. Similar to the proof of Lemma 4.6.2, denote by P the J-holomorphic biangle bounded by γ and the circle L_{R_C}, and by B the J-holomorphic triangle with boundary on $\eta^{\pm}$ and L_{R_C}. We have the geometric inequality

$$\kappa\int_P \omega > \kappa\int_B \omega,$$

and as in Lemma 4.6.2, we have that

$$\begin{aligned}
\kappa\int_P \omega &= \pi\tilde{\mu}(P) - \frac{1}{4}\int_{L_{R_C}} H_{L_{R_C}} \\
&= 2\pi - 2\tan^{-1}\left(\sqrt{\frac{(1+2r_1^2)^3}{r_1^4}\frac{R_C^4}{(1+2R_C^2)^3} - 1}\right) + \pi\left(\frac{R_C^2 - 1}{1+2R_C^2}\right) \\
&= \pi - 2\tan^{-1}\left(\sqrt{\frac{(1+2r_1^2)^3}{r_1^4}\frac{R_C^4}{(1+2R_C^2)^3} - 1}\right) + \pi R_C^2\left(\frac{3}{1+2R_C^2}\right)
\end{aligned}$$

On the other hand, we can estimate $\int_B \omega$ by the Euclidean area of B. Note that since $r < R_C < 1/\sqrt{2}$, we have that

$$\omega = \frac{2r}{(1+2r^2)^2}\,dr\,d\phi \geq \frac{2}{1+2R_C^2}\,r\,dr\,d\phi \geq \omega_0,$$

where ω_0 is the Euclidean area form. The Euclidean area of B is given by

$$\int_B \omega_0 = \frac{\pi R_C^2}{4} - \frac{r_1 R_C}{\sqrt{2}}.$$

Then the geometric inequality gives

$$\pi R_C^2 \left(\frac{3}{1+2R_C^2} - \frac{3}{2} \right) > \pi - 2\tan^{-1}\left(\sqrt{\frac{(1+2r_1^2)^3}{r_1^4} \frac{R_C^4}{(1+2R_C^2)^3}} - 1 \right) - 3\sqrt{2} r_1 R_C.$$

If R_C is bounded below by $\varepsilon > 0$, the right-hand side converges to 0 as $C \to \infty$ while the left-hand side is strictly less than 0, a contradiction. This completes the proof. □

4.7 Singularities for equivariant tori

To restrict the number and variety of singularities under consideration, we want to largely follow the ideas of Wood ([51], [52]), although we have to make many adjustments to account for the differences in the situations.

As Wood's work relies heavily on [33, Theorem B], we first prove a version of Neves' Theorem B for our situation. In fact, we show that we can modify the monotone version [34, Theorem B] (see Theorem 2.3.2) to apply to $\mathbb{CP}^2$.

Lemma 4.7.1. *Let L be a monotone Lagrangian mean curvature flow in $\mathbb{CP}^2$ with a finite-time singularity at $T < \infty$. For any sequence L_s^j of rescaled flows, the following property holds for all $R > 0$ and almost all $s < 0$:*

For any sequence of connected components Σ_j of $B_{4R}(0) \cap L_s^j$ that intersect $B_R(0)$, there exists a special Lagrangian cone Σ in $B_{2R}(0)$ with Lagrangian angle $\bar{\theta}$ such that, after passing to a subsequence,

$$\lim_{j\to\infty} \int_{\Sigma_j} f(\exp(i\theta_s^j))\phi \, d\mathscr{H}^2 = m f(\exp(i\bar{\theta}))\mu(\phi)$$

for every $f \in C(S^1)$ and every smooth ϕ compactly supported on $B_{2R}(0)$, where μ and m denote the Radon measure of the support of Σ and its multiplicity respectively.

Proof. By Proposition 3.3.4, a monotone torus in $\mathbb{CP}^2$ lifts to a monotone spherical Lagrangian 3-torus in $S^5 \subset \mathbb{C}^3$. The flow also lifts, becoming a flow with a singularity at a time $\tilde{T} < 1/2$ (recall that $t = 1/2$ is the singular time of the sphere S^5). The singularity in the lift occurs along an S^1, the Hopf fibre above the singular point of the original flow.

At this point we can already apply [34, Theorem A] to show that we have convergence to a finite set of special Lagrangian cones with angles θ_k. We want to show that we can instead apply [34, Theorem B], which a priori only applies in $\mathbb{C}^2$. Indeed, the only part of the proof of Theorem B which does not hold in higher dimensions is [34, Lemma 5.2]. So we have to show that for all j sufficiently large, there exists some $C > 0$ such that

$$\left(\mathscr{H}^3(A)\right)^{2/3} \leq C\mathscr{H}^2(\partial A), \tag{4.7.1}$$

for any open subset A of $L_s^j \cap B_{6R}(0)$ with rectifiable boundary. To do so, we have to hop between the original flow in $\mathbb{CP}^2$ and the lifted flow in $\mathbb{C}^3$. First, note that since the Hopf fibration is an isometry up to scale, the final time T gives an inequality relating the areas of subsets of the flows (note that no such inequality would exist if $T = \infty$ or equivalently $\tilde{T} = 1/2$). If A is an open subset of L_s^j and A_0 is its image under the Hopf fibration, we have that there exists constants $c_1(T), c_2(T)$ such that

$$c_1(T)\mathscr{H}^2(A_0) \leq \mathscr{H}^3(A) \leq c_2(T)\mathscr{H}^2(A_0).$$

Instead of using the Michael–Simon Sobolev inequality [32] as in the proof of [34, Lemma 5.2], we instead apply Brendle's Michael–Simon Sobolev inequality for Riemannian manifolds with positive curvature [7] to the flow in $\mathbb{CP}^2$. Denote by $A_0 \subset L$ the image of A under the Hopf fibration. Then, following the idea in [34, Lemma 5.2], we have

$$\begin{aligned}\left(\mathscr{H}^2(A_0)\right)^{1/2} &\leq C(R)\int_{A_0}|H_0|\,d\mathscr{H}^2 + C\mathscr{H}^1(\partial A_0)\\ &\leq C\left(\mathscr{H}^2(A_0)\right)^{1/2}\left(\int_{A_0}|H_0|^2\right)^{1/2} + C\mathscr{H}^1(\partial A_0).\end{aligned}$$

Neves's proof of Theorem A implies that

$$\lim_{j\to\infty}\int_{L_s^j\cap B_R(0)}|H|^2\,d\mathscr{H}^3 = 0$$

for almost all s and all $R > 0$, so we have that

$$\lim_{j\to\infty} \int_{\pi(L_s^j \cap B_R(0))} |H_0|^2 \, d\mathscr{H}^2 = 0$$

also. Then rearranging the above calculation we have that there is a universal constant C such that for all j sufficiently large,

$$\left(\mathscr{H}^2(A_0)\right)^{1/2} \leq C\mathscr{H}^1(\partial A_0).$$

Now using the Michael–Simon Sobolev inequality on the lifted flow in $\mathbb{C}^3$, we obtain

$$\begin{aligned}
\left(\mathscr{H}^3(A)\right)^{2/3} &\leq C\int_A |H| \, d\mathscr{H}^3 + C\mathscr{H}^2(\partial A) \\
&\leq C\left(\mathscr{H}^3(A)\right)^{1/2}\left(\int_A |H|^2 \, d\mathscr{H}^3\right)^{1/2} + C\mathscr{H}^2(\partial A) \\
&\leq C(T)\left(\mathscr{H}^2(A_0)\right)^{1/2}\left(\int_A |H|^2 \, d\mathscr{H}^3\right)^{1/2} + C\mathscr{H}^2(\partial A) \\
&\leq C(T)\mathscr{H}^1(\partial A_0)\left(\int_A |H|^2 \, d\mathscr{H}^3\right)^{1/2} + C\mathscr{H}^2(\partial A) \\
&\leq C(T)\mathscr{H}^2(\partial A)\left(1 + \left(\int_A |H|^2 \, d\mathscr{H}^3\right)^{1/2}\right),
\end{aligned}$$

and so the result (4.7.1) follows. The rest of the proof follows as in [34], and the result factors down to $\mathbb{CP}^2$ as desired. □

With Lemma 4.7.1 in hand, we can prove many of the results of Wood [52] for our situation.

Proposition 4.7.2. *Suppose L_γ is an equivariant monotone Clifford or Chekanov torus in $\mathbb{CP}^2$. Let $T \in (0, \infty]$ be the maximal existence time for L_γ.*

1. *If γ is initially embedded, it is embedded for all $t \in [0, T)$. If $L_{\gamma_1}, L_{\gamma_2}$ are two initial conditions with a finite number of intersections, then the number of intersections of γ_1 and γ_2 is a decreasing function in t. Similarly, the intersection number of γ with any cone C_b^a is also a decreasing function in t.*

2. *If L_γ has a finite-time singularity, then it must occur at the origin.*

3. *The type I blow-up of any singularity is the cone $C^0_{\pi/2}$.*

4. *The type II blow-up is a Lawlor neck asymptotic to the type I blow-up. The blow-up is independent of rescaling sequence.*

In addition, we also expect the following result of Wood [52] to hold in our case, though the proof certainly must differ due to the global topology of the situation. We do not directly use the result in the sequel, so we leave it as conjecture.

Conjecture 4.7.3. *The type I blow-up of any singularity is a multiplicity 1 copy of the cone $C^0_{\pi/2}$. The blow-up is independent of the rescaling sequence. (c.f. [52, Theorem 5.2.8])*

However, we do sketch a proof that the singularity is multiplicity 1 locally, in the following sense.

Proposition 4.7.4. *Any sequence of connected components as in the statement of Theorem 4.7.1 converges to a multiplicity 1 copy of the cone $C^0_{\pi/2}$.*

Proof of Proposition 4.7.2.

1. All 3 statements may be proven by variations on the same argument, which dates back to Angenent [2] applying a classical result of Sturm [45] on the zeroes of a uniformly parabolic PDE (see [2, Proposition 1.2]).

   We show the intersection of L_{γ_i} is decreasing, the other statements follow similar methods. Since L_{γ_i} is a Clifford or Chekanov torus, γ_i does not pass through the origin or infinity before the singular time. In particular, for any $T' < T$ we can find an annulus $A(r,R) \subset \mathbb{C}$ with $\gamma_i \subset A(r,R)$, for all $t \in [0,T']$. On this annulus, the evolution equation for γ_i is uniformly parabolic for both i.

   Then we are in the situation of [2, Theorem 1.3], and therefore the number of intersections of γ_1 and γ_2 is decreasing in t.

2. The statement is the same as [52, Theorem 5.2.6], but the proof in our case uses a method closer to that found in the proof of [52, Theorem 5.2.15].

For a contradiction, suppose $L_\gamma = F_\gamma(L)$ has a finite-time singularity not at the origin, i.e. at a point $x \in \mathbb{C}^*$. Since L_γ is monotone, the singularity is type II by Theorem 4.1.2. Consider then a type II blow-up sequence, that is to say a sequence of space-time points (p_i, t_i) with $x_i = F_\gamma(p_i) \to x$ as $i \to \infty$, where $t_i \in [0, T - 1/i]$, satisfying

$$|A_{t_i}(p_i)|^2 \left(T - \frac{1}{i} - t_i\right) = \max_{t \in [0, T-1/i], p \in L_\gamma} \left(|A_t(p)|^2 \left(T - \frac{1}{i} - t\right)\right).$$

Then we can find a subsequence with

(a) $|A_{t_i}(p_i)| \to \infty$ monotonically,

(b) $|A_{t_i}(p_i)|^2 \left(T - \frac{1}{i} - t\right) \to \infty$.

As in [52, Theorem 5.2.15], we have that the parabolic rescaling of F_γ

$$F_\tau^{(x_i, t_i)}(p) := A_i \left(F_{t_i + A_i^{-2}\tau}(p) - x_i\right)$$

with a factor $A_i = |A_{t_i}(p_i)|$ around (x_i, t_i) converges locally smoothly to a limiting eternal flow in $\mathbb{C}^2$, the type II blow-up L_τ. Since the finite-time singularity is not at the origin by assumption, the origin goes to infinity under the rescaling and hence the equivariance becomes a translational symmetry for the type II blow-up, i.e. there is a vector $V \in \mathbb{C}^2$ that is tangent to L_τ for all space-time points. Hence the type II blow-up is characterised by time-dependent curves $\gamma_\tau \in \mathbb{C}$ given by the intersection of L_τ with the orthogonal complement of $\operatorname{span}(V, JV) \subset \mathbb{C}^2$.

By definition of the rescaling, $|A|$ takes a value of 1 on the type II blow-up at the space-time point $(0,0) \in \mathbb{C}^2 \times (-\infty, \infty)$. Since L_τ is given by $\gamma_\tau \times \mathbb{R}$, the second fundamental form of L_τ is determined by the geodesic curvature of γ_τ. Therefore the geodesic curvature of γ_τ is non-zero at $(0,0)$, but this implies the mean curvature of L_τ is non-zero at $(0,0)$. So L_τ is not special Lagrangian.

However, we can use Lemma 4.7.1 in the same way as Wood uses Neves' Theorem B in the proof of [52, Theorem 5.2.13]. In particular, we find that we have convergence of the Lagrangian angle θ_τ^i of the type II rescaling to a constant $\bar{\theta}$

on any bounded parabolic region. But this implies L_τ is special Lagrangian, a contradiction.

3. The ideas used here are similar to those in the proof of [52, Theorem 5.2.8]. The main differences are as follows. We have monotone Lagrangians as opposed to almost-calibrated, which leads to additional possible singularities coming from the topology that Wood is able to rule out. However, we employ a stronger symmetry; the $\mathbb{Z}_2$-symmetry drastically restricts the possible planes that can occur in the blow-up. Furthermore, we are able to use Lemma 4.7.5, which is currently a positive curvature only fact, to eliminate some of the topological issues.

   By the above, the singularity must occur at the origin. By Lemma 4.7.1, the type I blow-up is a union of special Lagrangian planes, and furthermore each connected component must converge to a pair of planes with equal angle. Since L_γ is equivariant and does not pass through the origin, the only possible planes in the blow-up are $l_0, l_{\pi/2}$ (with Lagrangian angle 0 or π) and $l_{\pm\pi/4}$ (with Lagrangian angle $\pm\pi/2$). This follows since the $\mathbb{Z}_2$-symmetry leads to embeddedness breaking for any other limiting plane.

   We now claim that the $\mathbb{Z}_2$-symmetry prohibits the planes l_0 and $l_{\pi/2}$ from occurring. Suppose we have a sequence of rescalings $L^i_s = \lambda_i L_{\lambda_i^{-2}s}$, where we assume the singularity is at $T = 0$ for ease of notation. Let L^i_s be given by profile curves γ^i, where we have suppressed the s notation for convenience. Suppose further that γ^i converges to give the real axis in the limit. Then we can find a sequence of points $x^i \in \gamma^i$ with $x^i \to 1 \in \mathbb{C}$. Without loss of generality and using the $\mathbb{Z}_2$-symmetry across the axes, we can assume that x^i lie in the strictly real quadrant i.e. the region $Z = \{re^{i\phi} \in \mathbb{C} : \phi \in (0, \pi/2)\}$. Choose $R > 0$ such that $x_i \in B_{4R}(0)$ for i sufficiently large, and consider the sequence of connected components $\sigma^i \subset \gamma^i$ containing x^i, in the sense that σ^i satisfies the inclusions

$$\begin{array}{ccc} \sigma^i & \longrightarrow & \gamma^i \\ \downarrow & & \downarrow \\ \Sigma^i & \longrightarrow & L^i \end{array}$$

for the corresponding sequence of connected components Σ^i of L^i. For i sufficiently large, we have that σ_i intersects B_R, and hence Theorem 4.7.1 implies that we have convergence of the angle θ^i against test function to a constant multiple of 2π.

Suppose σ^i intersects the imaginary axis. Then by $\mathbb{Z}_2$-reflection, σ^i is the mirror image on the quadrant $-\bar{Z}$, i.e. the map $z \mapsto -\bar{z}$ preserves σ^i. Since θ can be written for a profile curve $\gamma(s)$ as

$$\theta(\gamma(s)) = \arg(\gamma(s)) + \arg(\gamma'(s)),$$

we have that

$$\theta\left(-\overline{\gamma(s)}\right) = \arg\left(-\overline{\gamma(s)}\right) + \arg\left(-\overline{\gamma'(s)}\right) = -\theta(\gamma(s)) \mod \pi,$$

since the orientation of $\gamma'(s)$ can be freely chosen. However, since the angle is continuous, and the angle where σ^i intersects the imaginary axis is $\pi/2$, we have that

$$\theta\left(-\overline{\sigma^i}\right) = -\theta(\sigma^i) + \pi \mod 2\pi. \tag{4.7.2}$$

Without loss of generality, suppose l_0 appears in the limit with angle 0 (angle π is similar). By Theorem 4.7.1, we have for any $\varepsilon > 0$ there exists an N such that for all $i > N$ the "ε-bad subset"

$$T^i = \{x \in \sigma^i \cap B_{2R}(0) : |e^{i\theta^i} - 1| \geq \varepsilon\}$$

has

$$\frac{\mathscr{H}^1(T^i)}{2R} < \varepsilon.$$

On the other hand, we can also find N such that the "ε-good subset"

$$S^i = A^i = \{x \in \sigma^i \cap B_{2R}(0) : |e^{i\theta^i} - 1| < \varepsilon\}$$

has for $i > N$

$$\frac{\mathscr{H}^1(S^i)}{2R} > 1/2$$

since the density in the limit is at least 1. (Here we have the results in [52, Section 5.1] to conclude that it suffices to consider the $\mathscr{H}^1$ measure of the curve γ rather than the $\mathscr{H}^2$ measure of the Lagrangian.) But this contradicts (4.7.2) since the same would also apply to $-\overline{T^i}$. So σ^i cannot intersect the imaginary axis in $B_{4R}(0)$.

Since σ^i does intersect the ball $B_R(0)$, we can find at least two distinct connected components σ_1^i and σ_2^i of $\sigma^i \cap A(R,2R)$, where $A(a,b)$ is the annulus

$$A(a,b) = B_b(0) - B_a(0).$$

Hence the density ratio

$$\lim_{i\to\infty} \frac{\mathscr{H}^1(\sigma^i \cap A(R,2R))}{R} \geq 2.$$

In addition, we can choose σ_1^i and σ_2^i such that they have different orientations, i.e. one must go from the outside to the inside and vice versa. Hence, they must converge to different limiting curves in the blow-up. There are only two possible limiting planes with angle 0, either the positive real axis oriented in the positive direction or the imaginary axis oriented in the negative direction. To see this, note that the possible limiting planes for curves in the quadrant Z are given by lines l_a with $a \in [0, \pi/2]$. The plane given by l_a has angle either $2a$ or $2a + \pi$, depending on the chosen orientation. So the only way to have angle 0 mod 2π is to have $a = 0$ or $\pi/2$, with the above stated orientations. Hence the type I blow-up therefore contains both l_0 and $l_{\pi/2}$.

Our goal now is to bound the density ratio above to show that

$$\lim_{i\to\infty} \frac{\mathscr{H}^1(\sigma^i \cap A(R,2R))}{R} < 3,$$

which will then imply that

$$\lim_{i\to\infty}\frac{\mathcal{H}^1(\sigma^i\cap A(R,2R))}{R}=2$$

since the limit is integer valued.

Let σ^i_k be any connected component of $\sigma^i\cap A(R,2R)$. Since σ^i_k is part of the connected component σ^i, the angle of σ^i_k converges to 0 and hence for any $\varepsilon,\delta>0$ we can find N sufficiently large so that for $i>N$ the ε-bad subset

$$T^i_k=\{x\in\sigma^i_k:|e^{i\theta^i_k}-1|\geq\varepsilon\}$$

has

$$\frac{\mathcal{H}^1(T^i_k)}{R}<\delta.$$

Huisken's monotonicity formula implies that

$$f_i(t)=\int_{\gamma^i}\left|H-\frac{X^\perp}{2t}\right|^2\rho(\cdot,t)\,ds$$

converges to 0 in $L^1_{\text{loc}}((-\infty,0])$, where X is the position vector and ρ is the backwards heat kernel at $(0,0)$. Hence we can find a subsequence of f_i converging point-wise almost everywhere to 0, and hence for almost every t, we can find a bound on the curvature on any fixed sized ball after rescaling:

$$\int_{\gamma^i\cap B_{2R}}|H|^2\,d\mu\leq C.$$

Since the equivariant part of the curvature is uniformly bounded away from 0 on the annulus $A(R,2R)$, see equation (4.4.6), this implies that

$$\int_{\sigma^i_k}\kappa^2\,d\mu\leq C(R).$$

Let p_1,p_2 be points in σ^i_k, and let $\tau(p_i)$ be the Euclidean tangent angle at p_i. Parametrising σ^i_k by arclength, the fundamental theorem of calculus and Hölder's inequality give

$$|\tau(p_2)-\tau(p_1)| = \left|\int_{p_1}^{p_2} \frac{\partial}{\partial s}\tau\, ds\right| = \left|\int_{p_1}^{p_2} \kappa\, d\mu\right| \leq \left(\mathrm{dist}_\gamma(p_1,p_2)\right)^{\frac{1}{2}} \left(\int_{\sigma_k^i} \kappa^2\, d\mu\right)^{\frac{1}{2}},$$

where $\mathrm{dist}_\gamma(p_1,p_2)$ is the intrinsic distance between p_1 and p_2 in γ. Thus

$$|\tau(p_2)-\tau(p_1)| \leq C(R)\left(\mathrm{dist}_\gamma(p_1,p_2)\right)^{\frac{1}{2}}.$$

Recall that the Lagrangian angle θ at p is given by

$$\theta(p) = \arg(p) + \tau(p).$$

Then we claim that the for any $\varepsilon > 0$, the ε-bad subset T_k^i is empty for sufficiently large i. Suppose for a given $\varepsilon > 0$, we can find a sequence of points p^i in T_k^i with $\theta(p^i) = 2\varepsilon$. For a given i and $\delta > 0$ to be determined, suppose that $x \in \sigma_k^i$ satisfies

$$\mathrm{dist}_\gamma(x, p^i) \leq R\delta.$$

The Lagrangian angle at x satisfies

$$\theta(p^i) - \delta - C(R)\delta^{\frac{1}{2}} \leq \theta(x) \leq \theta(p^i) + \delta + C(R)\delta^{\frac{1}{2}},$$

so by choosing δ sufficiently small, we have that $x \in T_k^i$ for all x. But for sufficiently large N, we have that for $i > N$,

$$\frac{\mathscr{H}^1(T_k^i)}{R} < \delta,$$

a contradiction. Since the initial choice of $\varepsilon > 0$ was arbitrary, we have proven the claim. As a corollary of the claim, we have that for any connected component σ_k^i and any $\varepsilon' > 0$, we can find N such that for $i > N$ we can write σ_k^i as a ε'-small C^1-normal graph over either an angle 0 Lawlor neck or the real or imaginary axes. This holds since Lawlor necks and planes through the origin are the only equivariant special Lagrangians, see [52][Lemma 5.1.5]. Furthermore, since some point in σ_k^i must converge to either the real or imaginary axis, we can guarantee that σ_k^i intersects the ball B_R for sufficiently large i. This holds since as any Lawlor necks distance to the axis on $A(R,2R)$, is proportional to their minimal distance to the origin.

Now we are in a position to show that

$$\lim_{i\to\infty} \frac{\mathscr{H}^1(\sigma^i \cap A(R,2R))}{R} = 2. \tag{4.7.3}$$

If there are only two connected components σ_k^i, this follows from calculating the maximum density of a Lawlor neck intersecting B_R. Writing a Lawlor neck with minimum distance to the origin R as

$$\eta(\phi) = \frac{R}{\sqrt{\sin 2s}} e^{i\phi},$$

see [52][Lemma 5.1.5], direct calculation yields that

$$\int_{\eta\cap A(R,2R)\cap Z} d\mu = R\left(\sqrt{15} - 2E\left(\frac{1}{4}(\pi - 2\cot^{-1}(\sqrt{15}))\middle| 2\right)\right) < \frac{14}{5}R,$$

where $E(x|m)$ is the elliptic integral of the second kind. Hence

$$\frac{\mathscr{H}^1(\sigma_k^i \cap A(R,2R))}{R} < \frac{3}{2},$$

for all i sufficiently large, and since the density in the limit must be an integer, we conclude that

$$\lim_{i\to\infty} \frac{\mathscr{H}^1(\sigma_k^i \cap A(R,2R))}{R} = 2,$$

as desired.

So it remains to verify there are only two connected components. This is a matter of simple topology, as illustrated in Figure 4.11. There is no way to have more than 2 connected components with the curve remaining embedded. Hence we have shown that (4.7.3) holds, and hence that the connected component σ^i contributes

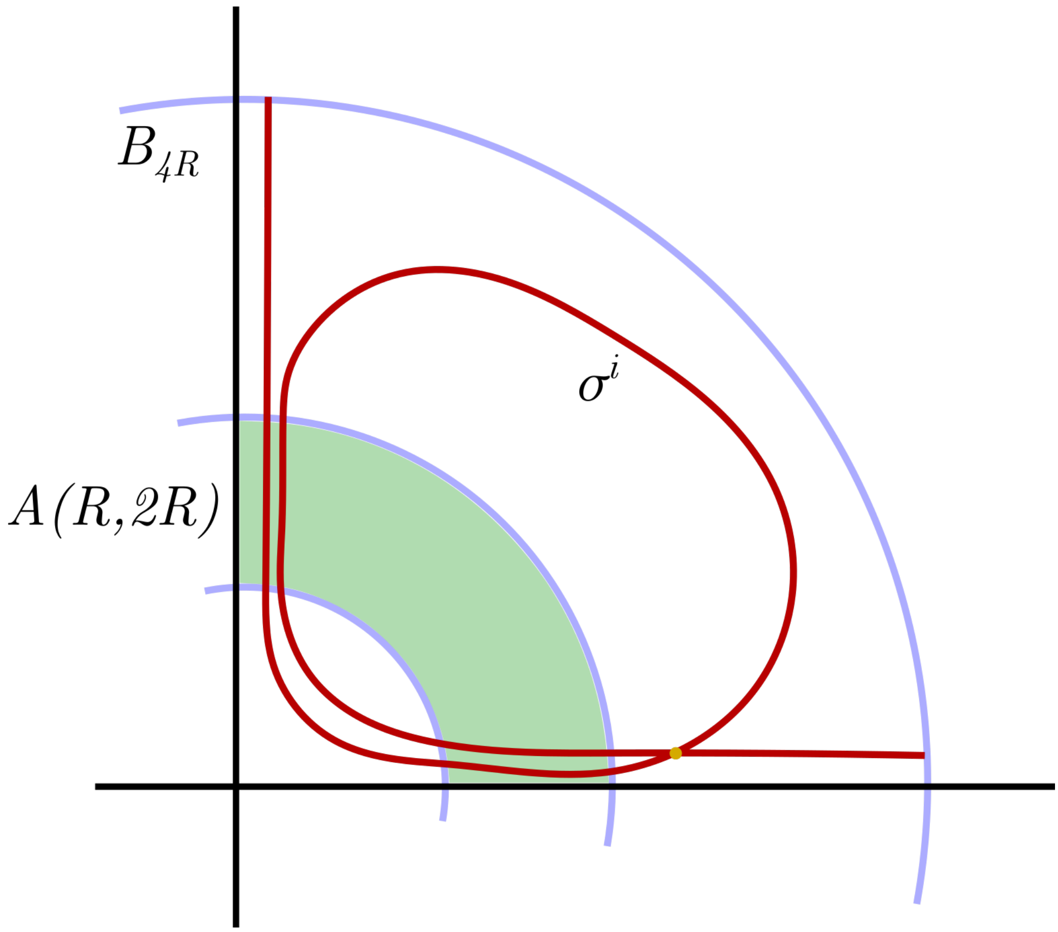

Figure 4.11: If $\sigma^i \cap A(R,2R)$ has more than two connected components, it is impossible to complete the curve σ^i without breaking embeddedness. The figure illustrates a failed attempt to do so.

to the type I blow-up a single copy of the real and imaginary axes, l_0 and $l_{\pi/2}$.

We can now derive the final contradiction. Leaving the quadrant Z, note that the $\mathbb{Z}_2$-symmetry implies that l_0 and $l_{\pi/2}$ both occur as a double density plane; in essence we have two Lawlor neck singularities forming with different angles, see Figure 4.12. Then we claim that the curves σ^i and $\bar{\sigma}^i$ must in fact intersect the real and imaginary axes within some small ball about the origin, and hence at this point either embeddedness breaks or the monotone condition breaks. This claim is proven after this proof since it is important in other contexts in this paper, see the Lemma 4.7.5. So we have derived a contradiction.

By equivariance, we must have that if $l_{\pi/4}$ appears in the blow-up then so does $l_{-\pi/4}$ and vice versa. Hence the blow-up is (a multiplicity n copy of) $C^0_{\pi/2}$.

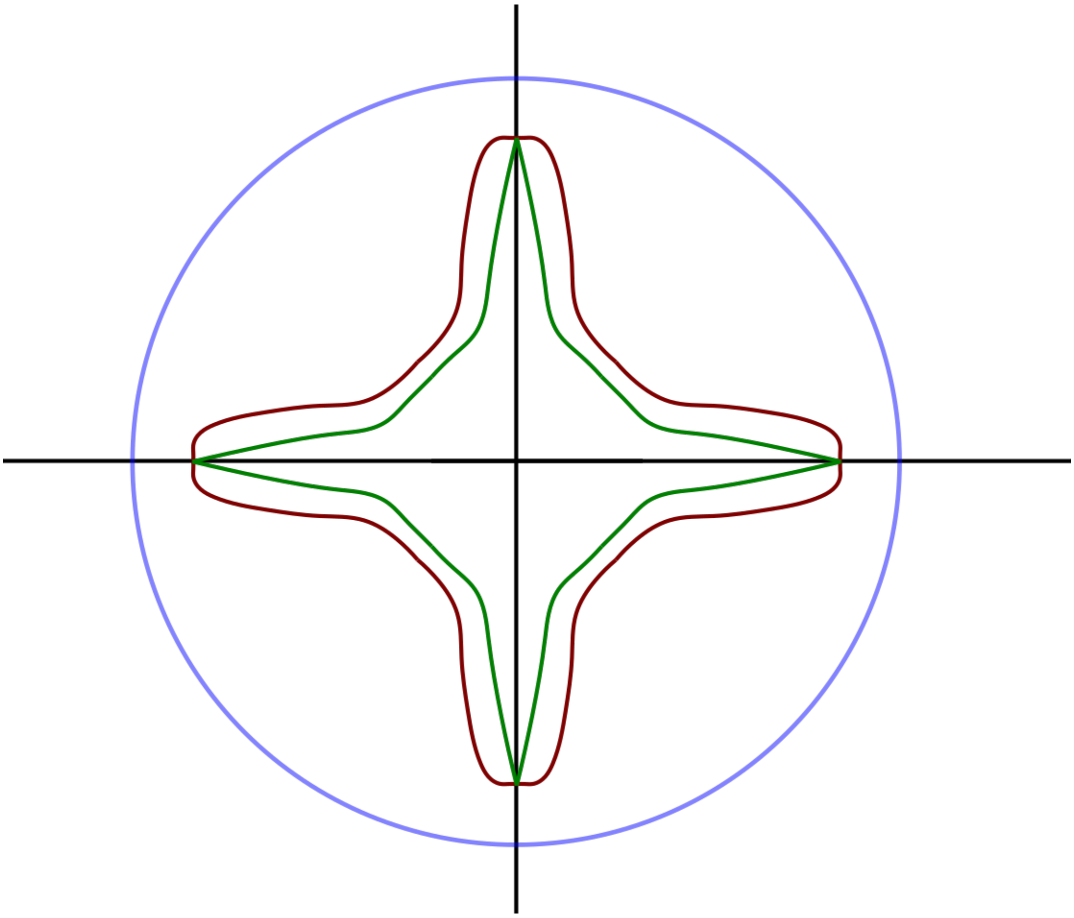

Figure 4.12: In the case of a double density copy of $C^{\pi/4}_{\pi/2}$, the scale lemma guarantees an intersection with the axes within some small ball. Thus we either have an non-embedded (green) or a non-monotone (red) Lagrangian, a contradiction.

4. See [52, Theorem 5.2.15]. In the above argument, we can already see how the rescaling is forming a Lawlor neck. The remaining details are similar to those used by Wood, and are hence omitted.

□

Sketch proof of Proposition 4.7.4. Let Σ_j be a sequence of connected components converging to a multiplicity 2 copy of the cone $C^0_{\pi/2}$. Since Σ_j are connected within a ball B_R of small radius R for sufficiently large j, and since they converge to a multiplicity 2 copy of the cone $C^0_{\pi/2}$, then within B_R, either Σ_j intersects the positive real axis greater than once, or Σ_j intersects both the real axis and the imaginary axis. Since they are connected within B_R and are equivariant, we must have that they bound a disc $D \subset B_R$. But $D \subset B_R$ implies $\kappa \int_D \omega < \kappa \int_{B_R} \omega << 2\pi$, so L is not monotone before the singular time, a contradiction. □

We conclude this section with the following lemma, which enables us to define surgery at singularities. The same proof but with the cone $C^{\pi/4}_{\pi/2}$ completes the proof of Proposition 4.7.2.

Lemma 4.7.5 (Scale lemma)**.** *Let L_γ be a monotone equivariant Lagrangian mean curvature flow in $\mathbb{CP}^2$ with a finite-time singularity at the origin at time $T < \infty$. Suppose the type I blow-up is the cone $C^0_{\pi/2}$ and the connected component converging to $C^0_{\pi/2}$ intersects the real axis.*

Then for any $R > 0$, $\varepsilon_0 > 0$, $\delta > 0$, there exists an ε with $0 < \varepsilon < \varepsilon_0$ and a time t' with $T-\delta < t' < T$ such that $L_\gamma \cap B_R$ intersects $C^0_{\pi/2+2\varepsilon}$ at the time t', where $B_R := B_R(0)$ is a ball of (Euclidean) radius R at the origin.

In essence, the scale lemma guarantees that singularity formation happens on an arbitrarily small scale. When we do surgery, this allows us to reduce the intersection number with the cone $C^0_{\pi/2}$, thus controlling the total number of surgeries any flow can undergo. Currently, no such result exists for Lagrangian mean curvature flow in Euclidean space. Indeed, the proof we give relies heavily upon barriers that cannot exist in Euclidean space; instead, all equivariant minimal surfaces that don't pass through the origin are Lawlor necks and are hence asymptotic to $C^0_{\pi/2}$ rather than intersecting it near the origin.

Proof. Suppose not. Then there exists $R > 0$, $\varepsilon_0 > 0$, $\delta > 0$ such that for any ε with $0 < \varepsilon < \varepsilon_0$ and t' with $T-\delta < t' < T$, $L_\gamma \cap B_R$ does not intersect $C^0_{\pi/2+2\varepsilon}$. Since T is the first singular time, we can, by taking a smaller $R > 0$ if necessary, have that $L_\gamma \cap B_R$ is empty at time $T-\delta$. We may also freely assume $R < 1$.

Note that since γ has a finite-time singularity at the origin, we have that there exists some $Q > 0$ such that $\max_{p\in\gamma} r(p) < Q$ for all time $t \in [0,T)$, where $r(p)$ is the Euclidean distance to the origin.

Consider the complete immersed minimal equivariant surfaces L_{γ_C} constructed in Theorem 4.6.3, where $\gamma_C(s) = r_C(s)e^{is}$ has initial values $r(0) = r_1, r'(0) = 0$. Since L_{γ_C}

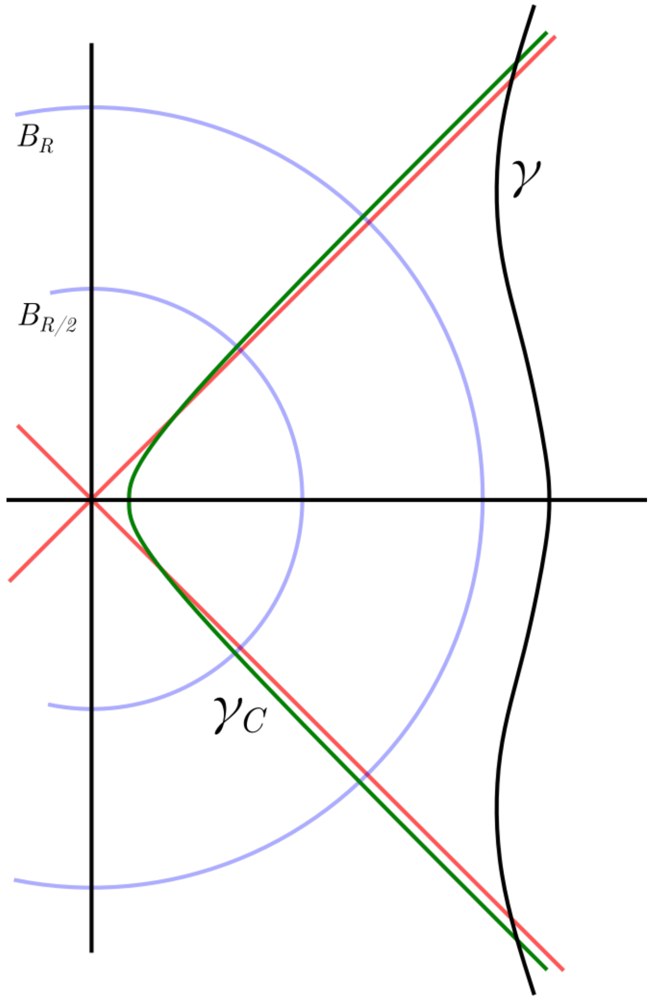

Figure 4.13: In order to form a singularity, γ must eventually intersect γ_C within $B_{R/2}$. But to do so, it must first intersect γ_C within the annulus $A(R/2,R)$, and hence intersects a cone $C^0_{\pi/2+2\varepsilon}$ for $\varepsilon > 0$.

is complete, γ_C intersects γ a finite number of times, non-increasing under the flow. By choosing $C > 27$ sufficiently large, we can find a curve γ_C such that

1. γ_C has maximum $r_2 > Q$.

2. $r_C(s) < R/2$ for all $s \in [-\pi/4, \pi/4]$, see Proposition 4.6.4.

3. The inner period ψ_C^- (see Lemma 4.6.2) satisfies $\psi_C^- < \pi/2 + 2\varepsilon_0$.

Consider the set

$$\mathscr{S} = \{p \in \gamma : p = \gamma_C(s) \text{ for } s \in [0, \psi_C/2]\},$$

where ψ_C is the period of γ_C. Since γ is monotone and both γ and γ_C are complete, this set is non-empty and finite for all time. Furthermore, if $p \in \mathscr{S}$, then $r(p) \in [R, Q]$ for all time by assumption, since otherwise we would have found a point $p \in L_\gamma \cap B_R$ intersecting $C^0_{\pi/2+2\varepsilon}$ for some ε with $0 < \varepsilon < \varepsilon_0$. Since the number of intersections between γ and γ_C is non-increasing, $L_\gamma \cap L_{\gamma_C} \cap B_R$ is empty for all time.

However, since the connected component giving $C^0_{\pi/2}$ in the blow-up contains the real axis, we have that the intersection with the real axis converges to 0 since otherwise the blow-up would have to contain a line C^0_a with $a < \pi/4$. In particular, there must exist some time $\tilde{t}$ with $T - \delta < \tilde{t} < T$ when $L_\gamma \cap L_{\gamma_C} \cap B_R$ is non-empty, a contradiction. □

Remark 4.7.6. The final step of the above proof can be simplified using the assumption that the Lawlor neck is the type II blow-up.

4.8 Graphical Clifford tori are stable

A natural condition to impose on solutions of the equivariant flow (4.4.6) is that γ is graphical over the minimal equivariant Clifford torus L_1. We have already studied minimal solutions of (4.4.6): the only embedded minimal solution is L_1. Thus, by Proposition 2.3.3, if we can prove that γ has long-time existence, then we obtain convergence to L_1 in infinite time.

Remark 4.8.1. Recall the fibration $\{L_\alpha\}$ by Clifford-type tori given by the moment map

$$\mu([x:y:z]) = \frac{1}{|x|^2+|y|^2+|z|^2}\left(|x|^2, |y|^2\right).$$

The equivariant fibres are

$$L_r = \{L_{re^{i\phi}} : r > 0\}$$

and from here on we denote by Ω the holomorphic volume form relative to L_r. Then for a Lagrangian L, we defined the Lagrangian angle θ relative to Ω by

$$\Omega_L = e^{i\theta}\,\mathrm{vol}_L\,.$$

Recall that in the Calabi–Yau case, if θ can be chosen to be a real-valued function, we call L zero-Maslov with respect to Ω. Furthermore, we call L almost-calibrated with respect to Ω if there exists some $\delta > 0$ such that

$$\cos\theta > \delta > 0.$$

Note that γ is graphical over L_1 if and only if γ is almost-calibrated with respect to Ω.

However, unlike in the Calabi–Yau case, θ defined in this way satisfies the evolution equation

$$\frac{\partial}{\partial t}\theta = \Delta\theta + d^{\dagger}\alpha,$$

and hence the parabolic maximum principle does not imply that $\cos\theta$ is increasing in time. Indeed, consider the following setup: let γ be a small ellipse with eccentricity $0 < e < 1$ centred on the origin. Then γ has a finite-time singularity at the origin. Furthermore, if e is sufficiently close to 1, the singularity is type II and has type II blow-up a Lawlor neck. In particular, there is no constant $\delta > 0$ such that $\cos\theta > \delta$ for all time. So almost-calibrated is not preserved in general.

Furthermore, even in the case where L_γ is monotone, we should not expect almost-calibrated to be preserved locally. Indeed, we construct an example later in the thesis where a non-graphical Clifford torus forms a finite-time singularity. However the construction seems to indicate that the almost-calibrated condition breaks locally in this case.

These examples illustrate two ideas. Firstly, we should consider γ as graphical rather than almost-calibrated. Secondly, we should only expect graphical to prevent type II singularities in the case that γ gives a monotone torus L_γ. This concludes the remark.

Proposition 4.8.2. *Let L_γ be a monotone equivariant Clifford torus, graphical over the minimal equivariant Clifford torus L_1. Then under mean curvature flow, L_γ exists for all time and converges to L_1 in infinite time.*

Proof. As mentioned above, it suffices to show that we have long-time existence. First, note that the graphical condition is preserved up to any potential singular time since the intersection number of γ with any cone C_b^a is decreasing in time by Proposition 4.7.2.

Suppose for a contradiction that we have a finite time singularity at time T. Any finite-time singularity must occur at the origin by Proposition 4.7.2, and must have blow-up given by $C^0_{\pi/2}$.

For any graphical γ and $\varepsilon > 0$, the cone $C^0_{\pi/2-2\varepsilon}$ intersects γ 4 times, dividing the Maslov 4 disc into 4 triangles. Denote the triangles intersecting the positive and negative real axes by P^+_ε and P^-_ε, and the triangles intersecting the positive and negative imaginary axes by Q^+_ε and Q^-_ε. Note that since L_γ is monotone,

$$\int_{P^+_\varepsilon+P^-_\varepsilon+Q^+_\varepsilon+Q^-_\varepsilon} \omega = 4\pi/6 = 2\pi/3.$$

Suppose the type II blow-up of a connected component is a Lawlor neck intersecting the real axis (this assumption is reasonable and simplifies the proof, but can be removed, see Remark 4.7.6). Then for any $\varepsilon > 0$, we have that

$$\int_{P^+_\varepsilon} \omega \to 0$$

as $t \to T$, where P_ε is the J-holomorphic triangle bounded by γ and $C^0_{\pi/2-2\varepsilon}$, intersecting the positive real axis and with a vertex at 0. But the total area contained outside the cone is bounded above by $\pi/2+2\varepsilon$, so

$$\int_{Q^+_\varepsilon+Q^-_\varepsilon} \omega < \pi/2+2\varepsilon.$$

Let $\varepsilon = \pi/48$. We can find a time t close to T such that

$$\int_{P^+_\varepsilon+P^-_\varepsilon} \omega < \pi/24.$$

Then at t,

$$\int_{P^+_\varepsilon+P^-_\varepsilon+Q^+_\varepsilon+Q^-_\varepsilon} \omega < \pi/2+\pi/12 = 7\pi/12 < 2\pi/3,$$

which contradicts L_γ being monotone. □

Remark 4.8.3. As in Remark 4.7.6, the assumption that the type II blow-up is a Lawlor neck simplifies the proof, but is not necessary.

Remark 4.8.4. The proof above actually relies on a secondary, weaker property of graphical Lagrangians. Instead of requiring a single pair of intersections with each line l_b, we can instead demand only that γ intersects the cone $C^0_{\pi/2}$ of opening angle $\pi/2$ in the minimum of 4 locations.

4.9 Equivariant Chekanov tori collapse

In the following, we will analyse the behaviour of equivariant Chekanov tori under mean curvature flow. Note first of all that any equivariant Chekanov torus does not intersect either the imaginary or real axis. Without loss of generality assume the former. Then there exists a cone C^0_ψ of maximal opening angle ψ such that $C^0_\psi \cap L_\gamma$ is non-empty. Since L_γ is monotone and the area inside the cone C^0_ψ is $\psi/2$, we must have that $\psi > 2\pi/3$.

We begin by proving there are no minimal equivariant Chekanov tori. This is interesting in its own right, and the method of proof suggests it may generalise to the non-equivariant case, a result that we conjecture in Chapter 5.

Proposition 4.9.1. *There is no minimal equivariant Chekanov torus.*

Proof. The result is immediate by the classification of equivariant tori in Section 4.6. However, we present a different proof since we believe the idea of the proof may be more widely applicable.

Let L_γ be an equivariant Chekanov torus. Without loss of generality, we are free to restrict to the subclass of Chekanov tori L_γ for which γ does not intersect the imaginary axis in $\mathbb{C}$.

Since γ does not intersect the imaginary axis and L_γ is equivariant, there is some maximal angle ψ such that C^0_ψ intersects γ. Since γ does not pass through the origin and does not intersect the imaginary axis, $\psi = \pi - \delta$ for some $\delta > 0$. Denote the first points of intersection of L_γ and C^0_ψ by p^+ and p^-, i.e. if $p \in C^0_\psi \cap L_\gamma$, then $r(p^\pm) \leq r(p)$.

Consider the J-holomorphic triangle P with boundary on $l_{\psi/2}, l_{-\psi/2}$ and γ with one vertex at the origin and the other two vertices at p^+ and p^-. As in Example 4.5.2,

$$\tilde{\mu}(P) = -\frac{1}{\pi}(\pi - 2\psi),$$

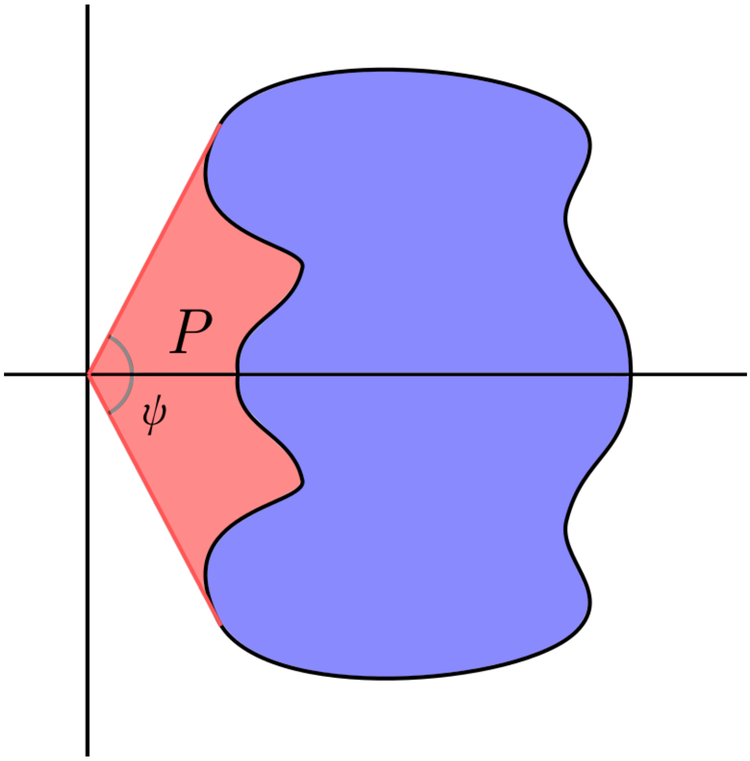

Figure 4.14: The triangle P collapses in finite time for a Chekanov torus.

and hence by (2.2.5) we have that

$$-\int_{\gamma_0} H = \kappa \int_P \omega + \pi - 2\psi,$$

where γ_0 is the arc of γ running from p^+ to p^-. But

$$\int_P \omega < \frac{\psi}{2} - \frac{\pi}{3}$$

since the area of P is bounded by the difference between the total area contained in the cone and the area of the Maslov 2 disc bounded by γ. So

$$-\int_{\gamma_0} H < 3\psi - 2\pi + \pi - 2\psi = \psi - \pi = -\delta < 0.$$

Hence L_γ is not minimal. □

Corollary 4.9.2. *Let L_γ be an equivariant Chekanov torus in $\mathbb{CP}^2$. Then under mean curvature flow, L_γ has a finite-time singularity at the origin* $[0:0:1]$.

Proof. By Proposition 4.9.1, there is no minimal equivariant representative in the Hamiltonian isotopy class of L_γ. Proposition 2.3.3 therefore implies that we have a finite-time singularity, which must occur at the origin by Proposition 4.7.2. $\square$

We also proffer an alternative proof using the evolution equation derived in Lemma 4.5.3.

Proof. Suppose there is no finite-time singularity. We are in the situation of Lemma 4.5.3, noting that the maximum opening angle must be a smooth function of t for all t sufficiently large. By the argument above, the right hand side of (4.5.4) is less than $-\delta$ for some $\delta > 0$. But then there is only a finite period of time when P has positive area, a contradiction. $\square$

4.10 Neck-to-neck surgery

In this section we analyse the behaviour of Lagrangian tori at the singular time. The equivariant condition necessarily (and intentionally) restricts us to two Hamiltonian isotopy classes of Lagrangian tori: the Clifford and Chekanov tori. We have shown in Proposition 4.9.2 that an equivariant Chekanov torus achieves a type II singularity at the origin with type II blow-up given by a Lawlor neck. Resolving the Lawlor neck singularity by neck-to-neck surgery gives a Clifford torus. We have also shown that almost-calibrated Clifford tori have long-time existence and convergence to the equivariant minimal Clifford torus. Our dream is that under Lagrangian mean curvature flow with surgeries, exotic tori in $\mathbb{CP}^2$ flow towards a minimal Clifford torus in infinite time, so we are led to ask the following question in our symmetric case:

Question 4.10.1. *Does an equivariant Chekanov torus flow to a minimal Clifford torus after neck-to-neck surgery?*

The answer to this question is yes, with the caveat that the number of neck-to-neck surgeries may be greater than one.

In fact, we will prove the following:

Theorem 4.10.2. *Let L_γ be a equivariant Clifford or Chekanov torus in $\mathbb{CP}^2$. Then, after a finite number of neck-to-neck surgeries, L_γ converges to the unique equivariant minimal Clifford torus L_1 in infinite time.*

Furthermore, we also give a proof of the following existence result:

Proposition 4.10.3. *Let $n \geq 0$ be an integer. Then*

1. *There exists a Clifford torus that undergoes exactly $2n$ neck-to-neck surgeries before converging to a minimal Clifford torus.*

2. *There exists a Chekanov torus that undergoes exactly $2n+1$ neck-to-neck surgeries before converging to a minimal Clifford torus.*

Definition 4.10.4. Let L_γ be an equivariant Lagrangian mean curvature flow with a finite-time singularity at $[0:0:1]$ at time $T < \infty$. Suppose the type I blow-up is the cone $C^0_{\pi/2}$ and the type II blow-up is (independent of rescaling) the Lawlor neck asympototic to $C^0_{\pi/2}$ intersecting the real axis. For any $r > 0$, we can find $\varepsilon > 0$ and a least time $t' < T$ such that $L_\gamma \cap B_r$ intersects $C^0_{\pi/2+2\varepsilon}$ as in Lemma 4.7.5. We define a new curve ζ in $\mathbb{C}$ which will give a Lagrangian L_ζ, which we will call the scale r surgery of L_γ.

Let $p^\pm = r' e^{\pm i(\pi/4+\varepsilon)}$ be the smallest radius points of intersection of $L_\gamma \cap B_r$ and $C^0_{\pi/2+2\varepsilon}$. Similar to the proof of Lemma 4.7.5, we can find a curve segment ζ' (intersecting the imaginary axis this time), smoothly tangent to $C^0_{\pi/2+\varepsilon}$ at points $q^\pm$, $-q^\pm$ at radius $r'' < r'$. Define ζ to be the union of ζ', $\gamma \cap A(r', \infty)$ and a smooth curve interpolating between $p^\pm$ and $q^\pm$.

Rescale radially and perform Moser's trick as in Vianna's construction to obtain from L_ζ a monotone surgery of L_γ.

Remark 4.10.5.

1. On the level of Lagrangians, this construction is not canonical. Since we need to use Moser's trick to obtain a monotone torus, monotone surgery is never going to be canonical unless you can flow directly through the singularity. In this case

however, there is a canonical way to perform Moser's trick since the equivariance means you can just rescale radially until you obtain a monotone torus. This is a quirk of the equivariance and cannot be expected in general.

2. The surgery procedure does not require that the type I blow-up is multiplicity 1 (even though we conjecture that all type I blow-ups in this situation are multiplicity 1). In the case that the multiplicity is higher than 1, the closest intersection point with the cone $C^0_{\pi/2+2\varepsilon}$ is continuous in time for times t sufficiently close to the singular time T. Hence there is no ambiguity about the neck to be cut and rotated in any case.

3. The surgery is canonical from a symplectic point of view since we always land in the same Hamiltonian isotopy class.

4. The surgery procedure is designed to reduce the intersection number of L_γ with $C^0_{\pi/2}$. Since we rescale radially, applying Moser's trick does not alter the intersection number.

Proof of Theorem 4.10.2. As in the proof of Proposition 4.9.2, we restrict to the case where L_γ intersects the imaginary axis either twice (in the case of a Clifford torus) or not at all (in the case of the Chekanov torus). Then by Proposition 4.7.2, equivariant tori can only achieve finite-time singularities at the origin $[0:0:1]$ with type I blow-up given by $C^0_{\pi/2}$, and type II blow-up given by a Lawlor neck asymptotic to $C^0_{\pi/2}$.

Since L_γ is compact, it has a finite number of intersections with $C^0_{\pi/2}$. By Proposition 4.7.2, the number of intersections is a decreasing function of time under mean curvature flow, and by definition neck-to-neck surgery decreases the number of intersections with $C^0_{\pi/2}$.

If L_γ is a Chekanov torus, Proposition 4.9.2 guarantees a finite-time singularity, at which point L_γ becomes a Clifford torus after surgery. Similar to the proof of Proposition 4.9.2, if L_γ is a Clifford torus intersecting $C^0_{\pi/2}$ greater than 4 times, then L_γ has inflection points and hence cannot be minimal. So L_γ either has a finite-time singularity, or after a finite time has only 4 intersections with $C^0_{\pi/2}$.

Since the number of intersections is strictly decreasing after surgery and both the above cases end in either surgery or a reduction of the number of intersections, after a finite time and a finite number of surgeries, we have the minimum number of intersections. It has already been shown in Proposition 4.8.2 and the succeeding Remark 4.8.4 that if L_γ has 4 intersections with $C^0_{\pi/2}$, then L_γ exists for all time and converges to L_1. The result follows. □

Remark 4.10.6. As in the secondary proof of Proposition 4.9.2 we could work with the evolution equations for $\int_P \omega$ directly, rather than just showing there are no minimal objects and applying Proposition 2.3.3.

We now construct a Clifford torus with a finite-time singularity to prove Proposition 4.10.3. The construction is somewhat technical but the idea is fairly simple: construct a monotone Clifford torus that has curvature sufficiently high in a neighbourhood of the origin, and use a barrier to stop γ from crossing the cone $C^0_{2\pi/3}$ for long enough that a singularity is inevitable. The constants chosen in the course of the proof are of no particular significance.

First, we prove a small lemma concerning the type I singular time of non-monotone Chekanov tori.

Lemma 4.10.7. *Let L_ζ be a non-monotone equivariant torus bounding a Maslov 2 disc D of area $A = \int_D \omega < \pi/3$. If ζ has $r > R(A) = \sqrt{A(\pi - 2A)^{-1}}$ everywhere, then L_ζ has a type I singularity away from the origin at time*

$$T = \frac{1}{6}\log\left(\frac{\pi}{3A-\pi}\right).$$

Proof. We have that

$$\frac{d}{dt}\int_D \omega = \kappa \int_D \omega - 2\pi,$$

so denoting $f(t) = \int_D \omega$, we have

$$f(t) = \left(A - \frac{\pi}{3}\right)e^{6t} + \frac{\pi}{3}.$$

Hence the final existence time T of L_γ satisfies

$$T \leq \frac{1}{6}\log\left(\frac{\pi}{3A-\pi}\right),$$

with equality if the singularity is type I.

Now consider $L_{R(A)}$ given by $\gamma_{R(A)}(s) = R(A)e^{is}$. Equation (4.4.1) reveals that $L_{R(A)}$ bounds a Maslov 4 disc of area $B = \pi\frac{2R(A)^2}{1+2R(A)^2}$, so $B > 2A$. Furthermore, a similar calculation to above gives the final existence time T' of $L_{R(A)}$ as

$$T' = \frac{1}{6}\log\left(\frac{2\pi}{3B-2\pi}\right).$$

Note that if L_ζ has a type II singularity, it must be at the origin and since $L_{R(A)}$ is a barrier to L_ζ, it must occur after T'. But $B > 2A$ implies that $T' > T$, and the result follows. $\square$

Proof of Proposition 4.10.3. We construct the $n = 1$ case, i.e. an equivariant Clifford torus L_γ with a finite-time singularity. The cases $n > 1$ follow an iterated version of the $n = 1$ case, and the Chekanov case follows automatically from the Clifford case.

See Figure 4.15 for the construction we now describe. Since γ is equivariant, it suffices to describe the construction only in the positive real quadrant, i.e. the region

$$Z = \{re^{i\phi} \in \mathbb{C} : \phi \in [0, \pi/2]\}.$$

Let γ be a equivariant curve such that L_γ is a Clifford-type torus and γ intersects the cone $C^0_{2\pi/3}$ 3 times in Z. Suppose γ has the minimum of 2 inflection points, i.e. points with $\langle \gamma, \nu \rangle = 0$. Note that then there are two biangles Q and R bounded by γ and $C^0_{2\pi/3}$. Furthermore, there is a triangle P formed by γ and the cone C^0_ψ where ψ is the widest opening angle such that C^0_ψ intersects γ exactly 2 times in Z.

We can choose γ such that we can find two Euclidean circles $\zeta \subset Q$ and $\eta \subset R$ each bounding discs with area $\pi/18$. Furthermore, we can choose γ such that ζ and η both have $r > R(\pi/18)$ as in the requirements of Lemma 4.10.7.

Furthermore, we can choose γ such that the triangle P has area $\pi/216$. It is clear we can make these choices whilst also choosing γ such that L_γ is monotone.

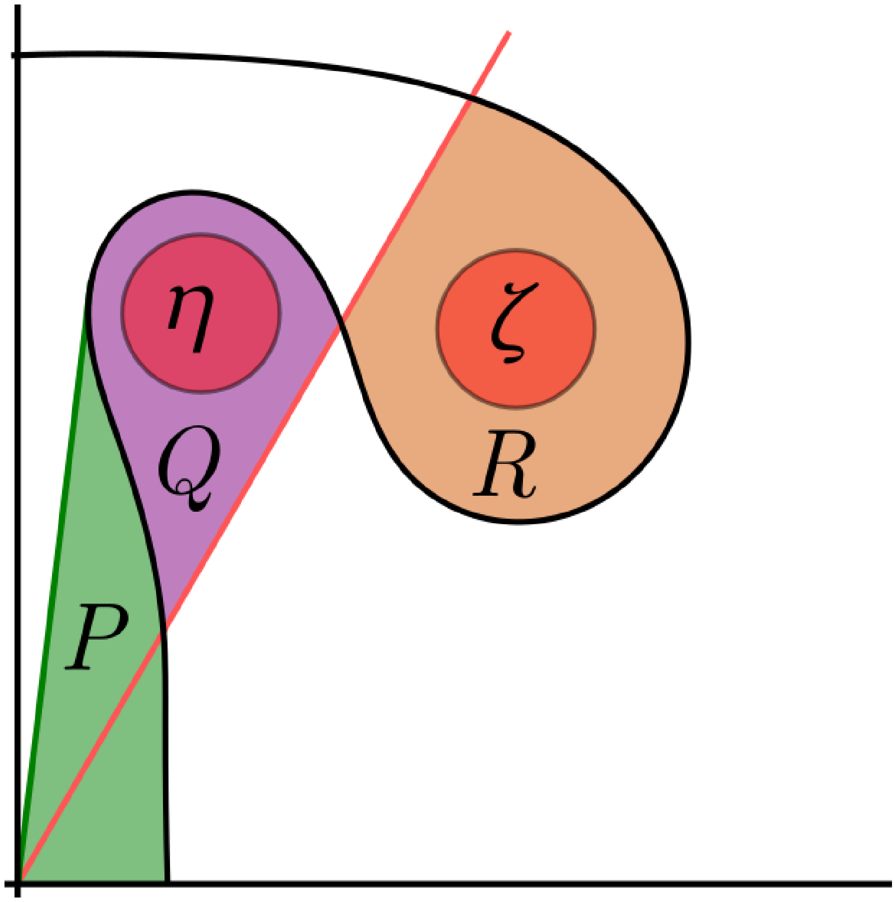

Figure 4.15: The construction of a Clifford torus with a finite-time singularity. Choosing the area of P sufficiently small and the areas of Q and R sufficiently large guarantees a finite-time singularity.

The non-monotone Chekanov tori L_ζ and L_η give lower bounds for the final time $\int_Q \omega, \int_R \omega > 0$ via direct application of Lemma 4.10.7. We have that that $\int_Q \omega > 0$ and $\int_R \omega > 0$ (and hence $\psi > 2\pi/3$) for all times $t < T$ with

$$T := \frac{1}{6}\log\left(\frac{6}{5}\right).$$

Suppose for a contradiction that the flow exists on the interval $[0,T]$. Then the triangle P exists for that period also, and the evolution of the triangle P is as in Lemma 4.5.3. We have that

$$\frac{d}{dt}\int_P \omega \leq \kappa \int_P \omega + (\pi - 2\psi) \leq \kappa \int_P \omega - \frac{\pi}{3},$$

since ψ is decreasing under the flow. Denoting $u(t) = \kappa \int_P \omega - \pi/3$, we see that $u(t)$ satisfies the differential inequality

$$\frac{d}{dt}u(t) \leq \kappa u(t),$$

to which we can apply Grönwall's inequality. Since $\kappa = 6$, we deduce that $u(t)$ satisfies

$$u(t) \leq u(0)e^{6t},$$

and hence

$$\int_P \omega \leq \left(\frac{\pi}{216} - \frac{\pi}{18}\right)e^{6t} + \frac{\pi}{18} = \left(1 - \frac{11}{12}e^{6t}\right)\frac{\pi}{18}$$

Hence the triangle P has a maximum existence time of

$$T' := \frac{1}{6}\log\left(\frac{12}{11}\right),$$

which is strictly less than T, a contradiction. Hence a finite-time singularity occurs in the period $[0, T]$. □

Chapter 5

A Thomas–Yau conjecture for the complex projective plane

We showed a Thomas–Yau type result holds for the equivariant tori in $\mathbb{CP}^2$: we have long-time existence with a finite number of surgeries and convergence to a minimal Lagrangian. In doing so, we defined a notion of neck-to-neck surgery, which to the author's knowledge is the first example of a Lagrangian mean curvature flow surgery in the literature. Here, it was vital that we were in a Fano manifold: there is not yet any analogue of the scale lemma (Lemma 4.7.5) in Calabi–Yau manifolds, and this lemma was vital in guaranteeing we had a finite number of surgeries.

From the point of view of the Fukaya category, the Clifford and Chekanov tori (and indeed all of Vianna's exotic tori) represent isomorphic objects, so we conjecture a full Thomas–Yau theorem for the complex projective plane:

Conjecture 5.0.1 (Thomas–Yau conjecture for the complex projective plane)**.** *Let L be an embedded monotone Lagrangian torus in $\mathbb{CP}^2$. Then L exists for all time under Lagrangian mean curvature flow with surgery, and converges in infinite time (after a finite number of surgeries) to a minimal Clifford torus.*

We also conjecture two related statements, which do not concern Lagrangian mean curvature flow directly, but are interesting independently.

Conjecture 5.0.2. *Let L be an embedded monotone Lagrangian torus in $\mathbb{CP}^2$. Then up to Hamiltonian isotopy, L is one of Vianna's exotic tori.*

Conjecture 5.0.3. *Let L be a minimal embedded Lagrangian torus in $\mathbb{CP}^2$. Then L is the standard monotone Clifford torus up to a rotation by* $\mathrm{PU}(3)$.

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